AD 691248

NOLTR 69-44

LINEAR THEORY OF BOTTOM REFLECTIONS

By
James R. Britt

12 MAY 1969

NOL

UNITED STATES NAVAL ORDNANCE LABORATORY, WHITE OAK, MARYLAND

NOLTR 69-44

ATTENTION

This document has been approved for public release and sale, its distribution is unlimited.

LINEAR THEORY OF BOTTOM REFLECTIONS

Prepared by:
James R. Britt

ABSTRACT: This paper gives a detailed mathematical development of a linear, spherical wave theory for the reflection of underwater explosion shock waves from plane bottoms of either fluid or rigid materials. The Laplace transform method of L. Cagniard is used to obtain integral solutions for the pressure which can be evaluated numerically.

The paper begins with linear equations of motion and proceeds in steps through the derivation of the wave equations and finally to solutions of the wave equations. Two methods of integrating the integral solution are discussed. First, the real part of the integral is separated from the imaginary part to allow integration using real arithmetic. Second, a method of integration using complex arithmetic is described. To aid the readers' understanding of the problem many important details are included in both the text and the appendices.

EXPLOSIONS RESEARCH DEPARTMENT
U. S. NAVAL ORDNANCE LABORATORY
WHITE OAK, MARYLAND

NOLTR 69-44

Linear Theory of Bottom Reflections 12 May 1969

This report is part of a continuing study of the interaction of the underwater explosion shock wave with the ocean bottom. This paper was written to clarify several questions arising in the mathematical development of the linear, spherical wave theory of bottom reflection which is presently being used. Many mathematical and physical details not usually included in the literature are included in this report to make the theory more accessible for the practical user.

This report was written in cooperation with and under the supervision of Dr. Hans G. Snay.

This study was supported by the Defense Atomic Support Agency through the Naval Ordnance Systems Command, ORD Task No. 035-200/092-1/UF17-553-502, Energy Focusing and Refraction Effects, Work Unit 1106, Subtask NA002.

The author is greatly indebted to R. W. O'Donnell who read the draft and made valuable suggestions.

E. F. SCHREITER
Captian, USN
Commander

C. J. ARONSON
By direction

TABLE OF CONTENTS

ILLUSTRATIONS

LIST OF SYMBOLS OCCURRING MOST FREQUENTLY

General Notation

D	depth of water
h	height of source above bottom
r	radial distance from axis
z	vertical distance, positive downward from the source
$R = (r^2 + z^2)^{1/2}$	slant distance from source to receiver
τ	time
P	excess pressure
P_d	free-water (direct) pressure pulse
$F(\tau)$	pulse shape of direct wave
$P_A = \frac{1}{R} H(\tau - R/c_1)$	spherical step wave pressure pulse
$H(\tau - R/c_1) = \begin{cases} 0 & \text{for } \tau \le R/c_1 \\ 1 & \text{for } \tau > R/c_1 \end{cases}$	Heaviside unit step function
c_1	sound velocity in water
c_2	sound velocity in liquid bottom
c_3	sound velocity in rigid bottom
c_4	propagation velocity of shear waves in a rigid bottom
ρ_1	density in water
ρ_2	density in bottom
$\vec{v}$	particle velocity vector
φ_1	velocity potential in water
φ_2	velocity potential in liquid bottom
P_1	excess pressure in water
P_2	excess pressure in bottom
$A_1(r,z,\tau)$	step wave pressure response in water

General Notation

$A_s(r,z,\tau)$	step wave pressure response in bottom
$-$ (over a symbol)	Laplace transform except in Chapters II and IV
$\sim$ (over a symbol)	complex conjugate
s	Laplace transform variable
λ	separation variable
$J_0(\lambda r)$	Bessel function of the first kind and of order zero
u, w	integration variables
Im	imaginary part of a complex quantity
Re	real part of a complex quantity
$\beta_i = (s^2 c_i^{-2} + \lambda^2)^{1/2}$	
K	reflection coefficient
$b = \rho_1/\rho_2$	ratio of density of water to density of bottom
n	number of times a wave has been reflected from the bottom
m	one of four waves for each n
${}_nP_m$	step wave pressure response for wave (n,m)
${}_nd_m$	height from receiver to image source (n,m)
${}_nR_m = (r^2 + {}_nd_m^2)^{1/2}$	slant distance from receiver to image source (n,m)
$\alpha_i = (u^2 + c_i^{-2})^{1/2}$	
${}_n\delta_m$	arrival time of the first response for wave (n,m)
$u_1 = {}_nR_m^{-2}\left[-i\tau r + {}_nd_m(\tau^2 - c_1^{-2}\ {}_nR_m^2)^{1/2}\right]$	a limit of integration
$\gamma = \left[u^2 r^2 + (\tau - {}_nd_m\,\alpha_1)^2\right]^{1/2}$	

General Notation

${}_nP'_m$	integrand of ${}_nP_m$

ROSENBAUM NOTATION USED IN CHAPTERS II AND IV

$R_m = {}_nR_m$	
$d_m = {}_nd_m$	
$\delta = {}_n\delta_m$	
$\delta_m = \delta/R_m$	
$\tau_m = \tau/R_m$	
$\ell_m = \arccos\,(d_m/R_m)$	angle of incidence of wave (n,m)
x	imaginary part of u when u is pure imaginary
$\omega = (c_1^{-2} - x^2)^{1/2}$	
$\overline{\omega} = (x^2 - c_1^{-2})^{1/2}$	
$M_m = \tau_m \cos \ell_m + (c_1^{-2} - \tau_m^2)^{1/2} \sin \ell_m$	
$N_m = \tau_m \cos \ell_m - (c_1^{-2} - \tau_m^2)^{1/2} \sin \ell_m$	
$K_m = \tau_m \cos \ell_m$	
$L_m = (\tau_m^2 - c_1^{-2}) \sin^2 \ell_m$	
ψ	integration variable for precursor defined by $\omega = (\sigma + M_m)/2 + \left[(\sigma - M_m)/2\right]\sin \frac{\pi\psi}{2}$
$D_m = \tau_m^2 \cos 2\ell_m + c_1^{-2} \sin^2 \ell_m$	
$E_m = 4\,(\sin^2 \ell_m \cos^2 \ell_m)\, \tau_m^2\, (\tau_m^2 - c_1^{-2})$	
$F_m = \tau_m^2 - c_1^{-2} \sin^2 \ell_m$	
γ_1	γ for positive real α_1
γ_2	γ for negative real α_1
$\overline{\gamma}_1$	γ for positive imaginary α_1
$\overline{\gamma}_2$	γ for negative imaginary α_1
F_∞	residue of singularity of ${}_nP'_m$ at infinity

NOTATION USED ONLY IN CHAPTER II

$\sigma = (c_1^{-2} - c_2^{-2})^{1/2}$

$\bar{\sigma} = (c_2^{-2} - c_1^{-2})^{1/2}$

K — reflection coefficient for real α_1 and imaginary α_2

$\bar{K}$ — reflection coefficient for imaginary α_1 and real α_2

NOTATION USED IN CHAPTER IV

$\sigma_1 = (c_1^{-2} - c_3^{-2})^{1/2}$

$\sigma_2 = (|c_4^{-2} - c_1^{-2}|)^{1/2}$

$A = \omega(c_4^{-2}/2 - c_1^{-2} + \omega^2)^2$

$B = \omega(c_1^{-2} - \omega^2)(\sigma_1{}^2 - \omega^2)^{1/2}(\sigma_2{}^2 + \omega^2)^{1/2}$

$C = \frac{b}{4} c_4^{-4} (\sigma_1{}^2 - \omega^2)^{1/2}$

$B_2 = \omega(c_1^{-2} - \omega^2)(\sigma_1{}^2 - \omega^2)^{1/2}(|\omega^2 - \sigma_2{}^2|)^{1/2}$

$\bar{A} = \bar{\omega}\, [c_4^{-2}/2 - c_1^{-2} - \bar{\omega}^2]^2$

$\bar{B} = \bar{\omega}\, (c_1^{-2} + \bar{\omega}^2)(\sigma_1{}^2 + \bar{\omega}^2)^{1/2}(\sigma_2{}^2 - \bar{\omega}^2)^{1/2}$

$\bar{C} = \frac{bc_4^{-4}}{4} (\sigma_1{}^2 + \bar{\omega}^2)^{1/2}$

$k = 1/c_{ST}$ — reciprocal of Stonley wave propagation velocity

K — reflection coefficient for real α_1 and α_4 and imaginary α_3

$\bar{\bar{K}}$ — reflection coefficient for real α_1 and imaginary α_3 and α_4

$\bar{K}$ — reflection coefficient for imaginary α_1 and α_3 and real α_4

R_{ik}, R_{-ik} — Stonley pole residues

$g_i = (k^2 - c_i^{-2})^{1/2}$

$a = \tau_m{}^2 - (k^2 - c_1^{-2} \cos^2 \ell_m)$

$f = 4\, \tau_m^2\, g_1{}^2 \cos^2 \ell_m$

$\psi_1 = \frac{2}{\pi} \arcsin \left[\frac{2\sigma_2 - \sigma_1 - M_m}{\sigma_1 - M_m}\right]$ — a limit of integration for $\tau < c_1^{-1} R_m$ and $c_3 > c_4 > c_1$

General Notation

${}_1\Delta_m$	contribution to ${}_1P_m$ from the Stonley pole resides

NOTATION USED ONLY IN CHAPTER III

ϕ	displacement potential in water
u_r	radial component of displacement
u_z	vertical component of displacement
λ'	Lamé's elastic constant of normal stress
μ	Lamés's elastic constant of shearing stress
T_{zz}	component of stress normal to water-bottom interface
T_{rz}	component of stress tangent to water-bottom interface
ψ, U	potentials used to express displacement and stress in a rigid bottom.
ξ	potential related to U by $U = -\partial\xi/\partial r$
$J_1(\lambda r)$	Bessel function of the first kind and of order one
$J_2(\lambda r)$	Bessel function of the first kind and of order two

NOTATION USED IN CHAPTER V

$x = Re(u)$	
$y = Im(u)$	
y_1	lower limit of integration defined in paragraph 5-1
y_2	upper limit of integration defined in paragraph 5-1
$\omega_1 = [c_1^{-2} + (x + iy_1)^2]^{1/2}$	lower limit of integration
$\omega_2 = [c_1^{-2} + (x + iy_2)^2]^{1/2}$	upper limit of integration

LINEAR THEORY OF BOTTOM REFLECTIONS

INTRODUCTION

The reflection of underwater explosion shock waves from the bottom of the sea is an interesting, but difficult problem. The mathematical model used today employs the acoustic approximation, namely the assumption of small amplitudes and a constant propagation velocity. This approximation permits the use of the linear (acoustic) wave equation which has implicit solutions. Certainly, shock waves from explosions are high amplitude (non-linear) waves, and a strict treatment requires an entirely different approach.

The accuracy of the acoustic approximation can, however, be improved somewhat by replacing the sonic velocity of the media with a correspondingly higher velocity which accounts for the increase of the actual propagation velocity with pressure. In the same way the decrease of the pressure with distance can be calculated for a non-linear rather than acoustic wave. Although these linearizations used with the acoustic method are crude approximations for high pressures, much can be learned about the salient features of the reflection process.

The calculation of the reflection of a plane, exponential wave from a homogeneous, plane bottom is relatively simple (Arons and Yennie [1]). Since this treatment has shortcomings, such as the prediction of an infintely long precursor wave, solutions for spherical waves have become of interest. The introduction of spherical waves considerably

complicates the problem. Even the highly simplified problem of the reflection of a solitary spherical, acoustic wave has a very involved solution.

One of the best treatments of the spherical wave is given by J. H. Rosenbaum [2]. In his Naval Ordnance Laboratory report Rosenbaum uses the method of L. Cagniard [3] to obtain integral solutions which are suitable for numerical calculations. Rosenbaum's report is concise and accurate, but is not very explicity written. It leaves the reader, especially a beginner in the field, with many unanswered questions. The paper is also out of print and not generally available.

The present paper has been written for the benefit of the practical user who needs a good understanding of the methods involved in the derivation of these equations to intelligently use and manipulate them and be able to attack more complicated problems.

The line of attack used in this paper is close to that of Rosenbaum, but many details omitted by him are included here. Rosenbaum's notation is used, whenever practical, to allow comparison with his paper. Additional equations are derived, and a brief discussion is given on the use of complex computer arithmetic in evaluating the integral solution.

CHAPTER I

SOLUTION OF THE LIQUID BOTTOM WAVE EQUATION

The purpose of this Chapter is to develop from basic hydrodynamic principles the acoustic wave equation in a fluid and obtain, using the method of Cagniard, an integral solution for the pressure from an underwater explosion over a liquid bottom.

1-1. GEOMETRY AND HYDRODYNAMIC RELATIONS

It is assumed that a spherical acoustic wave propagating through a layer of homogeneous fluid (water) of thickness D interacts with a bottom consisting of the lower half space of either homogeneous, non-viscous fluid or homogeneous, isotropic, and perfectly elastic solid. Because of the axial symmetry, a cylindrical-polar coordinate system is chosen with the source of the wave at the origin $(r,z,\phi) = (0,0,0)$. Being rotationally symmetric, the problem is independent of the polar angle ϕ. The task is to find the pressure at a receiver with coordinates (r,z) as shown in Figure 1-1.1. The source is at a distance h above the bottom and emits a free-water (direct) pressure pulse given by

$$\begin{aligned} P_d &= 0 && \text{for } \tau \le R/c_1 \\ P_d &= F(\tau - R/c_1)/R && \text{for } \tau > R/c_1 \end{aligned} \qquad (1\text{-}1.1)$$

The time τ is measured from the instant of explosion $\tau = 0$. The slant distance from source to receiver is $R = (r^2 + z^2)^{1/2}$, and the sound velocity in the water is c_1.

The simplest case, that for a liquid bottom, will be treated first. The more complicated equations for a bottom with shear strength (rigid bottom) will be derived later in Chapter III.

We assume that the water and the liquid bottom are ideal fluids in the sense that effects of viscosity and heat conduction may be neglected. We also assume that the waves are of small amplitude, that is, acoustic waves.

The equation of motion of a ideal, homogeneous fluid not affected by any external forces is then

$$\frac{\partial \vec{v}}{\partial \tau} + (\vec{v}\cdot\vec{\nabla})\,\vec{v} = -\frac{1}{\rho}\,\vec{\nabla}\,P, \tag{1-1.2}$$

where P is the excess pressure, $\vec{v}$ is the particle velocity vector, and ρ is the density of the fluid. For irrotational flow the particle velocity $\vec{v}$ can be expressed by $\vec{v} = -\vec{\nabla}\varphi$, where φ is the scalar velocity potential. Equation (1-1.2) can then be written

$$-\frac{\partial}{\partial \tau}(\vec{\nabla}\varphi) + \frac{1}{2}\,\vec{\nabla}\,v^2 = -\frac{1}{\rho}\,\vec{\nabla}P. \tag{1-1.3}$$

Space and time differentiation are interchangeable and the above equation can therefore be integrated with respect to the space variables. This gives the Bernoulli Equation which in the absence of external forces reads

$$\frac{P}{\rho_0} + \frac{1}{2}\,v^2 = \frac{\partial \varphi}{\partial \tau}\,, \tag{1-1.4}$$

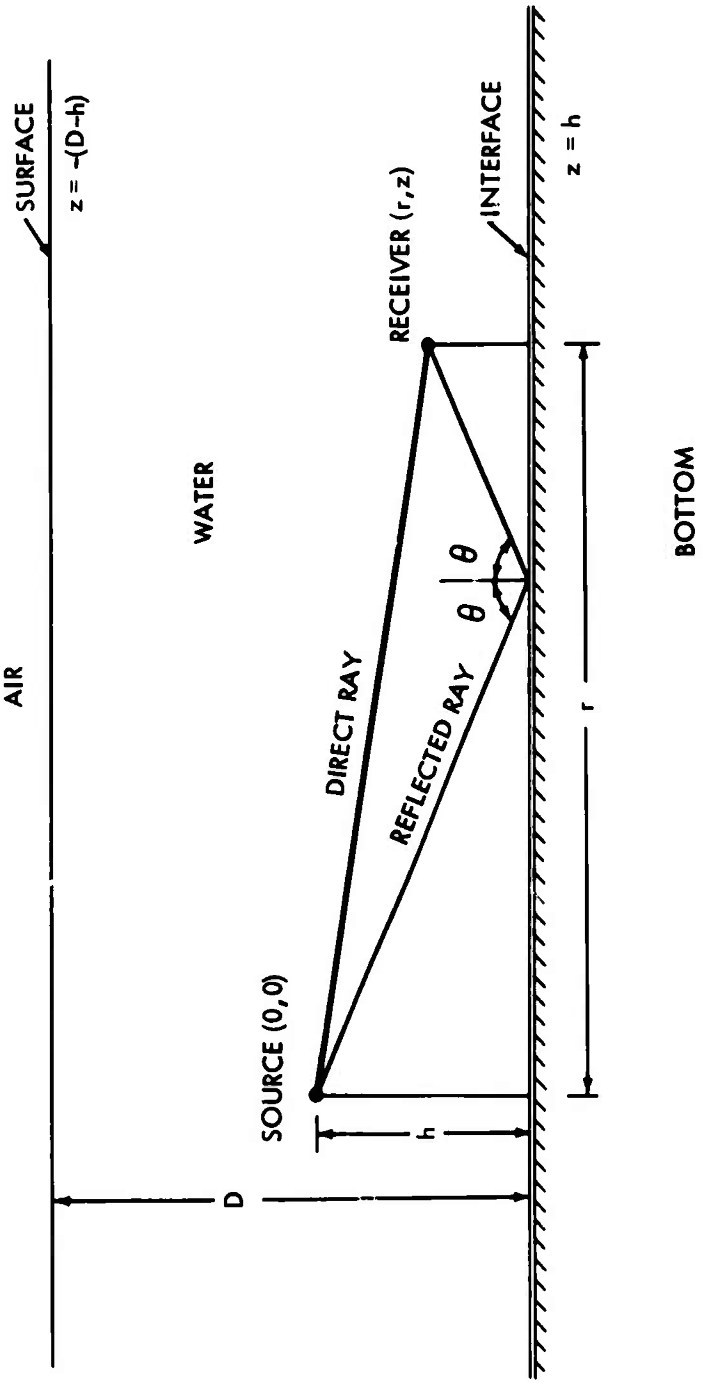

FIG. 1-1.1 GEOMETRY FOR ACOUSTIC PROBLEM

where we have assumed that the changes of ρ relative to its initial value ρ_0 are insignificant. We also have assumed that the fluid at infinity is at rest and has zero excess pressure.

In the acoustic approximation the particle velocity v is small compared to the sound velocity c. Since the term P/ρ is of the order of magnitude vc, the term $v^2/2$ is small compared to P/ρ and may be omitted to give the equation for the pressure

$$P = \rho_0 \frac{\partial \varphi}{\partial \tau} . \tag{1-1.5}$$

The equation of continuity for ideal fluids is

$$\frac{\partial \rho}{\partial \tau} + \rho_0 \vec{\nabla} \cdot \vec{v} = 0 , \tag{1-1.6}$$

where we have again assumed that the changes of ρ relative to its initial value ρ_0 are insignificant. Further we neglect all dissipative processes. This means an element of fluid cannot exchange heat with its neighbors and that no energy dissipation occurs. Hence changes in the physical state of the element take place at constant entropy, and we write

$$\frac{\partial P}{\partial \tau} = \left(\frac{dP}{d\rho}\right)_S \frac{\partial \rho}{\partial \tau} , \tag{1-1.7}$$

where S denotes changes at the constant entropy of the undisturbed fluid. The expression $\sqrt{(dP/d\rho)_S}$ is called the sound velocity and is denoted by c. In the acoustic approximation changes of c relative to the initial value c_0 are ignored.

Combining equations (1-1.6) and (1-1.7), we obtain

$$-\rho_0 \vec{\nabla}\cdot\vec{v} = \frac{1}{C_0^2}\frac{\partial P}{\partial \tau} \tag{1-1.8}$$

Substituting $\vec{v}$ and P in terms of the velocity potential yields the acoustic wave equation:

$$\nabla^2 \varphi = \frac{1}{C_0^2}\frac{\partial^2 \varphi}{\partial \tau^2} \ . \tag{1-1.9}$$

We now seek solutions of the acoustic wave equation which, with the aid of the Bernoulli Equation (equation (1-1.5)), can be used to construct the pressure-time history of a low amplitude underwater explosion shock wave.

1-2 WAVE EQUATIONS FOR WATER AND BOTTOM, BOUNDARY CONDITIONS

It has been assumed that the motion in the water and in the bottom can be represented by velocity potentials φ_1 and φ_2, the subscripts $_1$ and $_2$ will be used generally to denote, respectively, quantities of the water and of the bottom. In cylindrical coordinates φ_1 and φ_2 satisfy wave equations as follows:

in the water $$\frac{\partial^2 \varphi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi_1}{\partial r} + \frac{\partial^2 \varphi_1}{\partial z^2} = \frac{1}{c_1^2}\frac{\partial^2 \varphi_1}{\partial \tau^2} \tag{1-2.1}$$

in the bottom $$\frac{\partial^2 \varphi_2}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi_2}{\partial r} + \frac{\partial^2 \varphi_2}{\partial z^2} = \frac{1}{c_2^2}\frac{\partial^2 \varphi_2}{\partial \tau^2} \ , \tag{1-2.2}$$

where c_1 is the sound velocity in the water and c_2 is the sound velocity in the liquid bottom. The particle velocities are then

$$\vec{v}_i = -\vec{\nabla}\,\varphi_i \;, \; i = 1,\, 2. \qquad (1\text{-}2.3)$$

To comply with the physical picture of our problem, the following boundary conditions must be imposed:

1(a). At the surface of the water, $z = -(D - h)$, the excess pressure P_1 must vanish since we consider the water-air interface to be a free surface.

2(a). Near the source of the explosion the effects of the surface and the bottom do not contribute. The pressure is then that of the direct wave P_d given by equation (1-1.1) as R approaches zero.

3(a). At the bottom-water interface pressure and normal velocity must change continuously from one medium to the other. This requires $P_1 = P_2$ and $\partial\varphi_1/\partial z = \partial\varphi_2/\partial z$ for $z = h$. Hence differentiating with respect to time, we have

$$\frac{1}{\rho_1}\,\frac{\partial P_1}{\partial z} = \frac{1}{\rho_2}\,\frac{\partial P_2}{\partial z} \;. \qquad (1\text{-}2.4)$$

4(a). The disturbance must vanish for $z \to \infty$ so that P_2 approaches zero as z goes to infinity.

1-3 LAPLACE TRANSFORMATION

To solve the wave equations we use the Laplace transformation method. The Laplace transform $\bar{f}(r,z,s)$ with respect to time τ of any function $f(r,z,\tau)$ is defined

$$\bar{f}(r,z,s) = \int_0^\infty f(r,z,\tau)\,\exp\,(-s\tau)\;d\tau \;, \qquad (1\text{-}3.1)$$

where s is real and is the transform variable replacing the time τ.

Taking the transforms of equations (1-2.1) and (1-2.2) and noting that $F(\tau - R/c_1)$ is zero for $\tau \leq R/c_1$, we get

$$\frac{\partial^2 \bar{\varphi}_i}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\varphi}_i}{\partial r} + \frac{\partial^2 \bar{\varphi}_i}{\partial z^2} - \frac{s^2}{c_i^2}\bar{\varphi}_i = 0 \text{ , } i = 1, 2. \qquad (1\text{-}3.2)$$

Similarly the transform of the pressure $P_i = \rho_i(\partial\varphi_i/\partial\tau)$ is

$$\bar{P}_i = s\rho_i \bar{\varphi}_i, \; i = 1, 2. \qquad (1\text{-}3.3)$$

Substitution of equation (1-3.3) into equation (1-3.2) yields

$$\frac{\partial^2 \bar{P}_i}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{P}_i}{\partial r} + \frac{\partial^2 \bar{P}_i}{\partial z^2} - \frac{s^2}{c_i^2}\bar{P}_i = 0, \; i = 1, 2. \qquad (1\text{-}3.4)$$

The transformed boundary conditions are as follows:

1(b) At $z = -(D - h)$, the water surface, we have $\bar{P}_i = 0$.

2(b) Near the origin $\bar{P}_1$ approaches $\bar{P}_d = \exp(-sR/c_1)\,\bar{F}(s)/R$, where $\bar{F}(s)$ denotes the Laplace transform of $F(\tau)$.

3(b) At $z = h$, the bottom-water interface, we have $\bar{P}_1 = \bar{P}_2$ and $(\partial\bar{P}_1/\partial z)/\rho_1 = (\partial\bar{P}_2/\partial z)/\rho_2$.

4(b) As z tends to infinity $\bar{P}_2$ becomes zero.

1-4 SOLUTION OF THE LAPLACE TRANSFORMED WAVE EQUATIONS

Subject to the boundary conditions imposed on $\bar{P}_i$ above, the wave equation (1-3.2) can be solved by the method of separation of variables. Assuming a solution

$$\bar{P}_i(r,z,s) = R_i(r)\, Z_i(z), \qquad (1\text{-}4.1)$$

equation (1-3.2) can then be written

$$\frac{R_i''}{R_i} + \frac{1}{r}\frac{R_i'}{R_i} = \frac{s^2}{c_i^2} - \frac{Z_i''}{Z_i} \qquad (1\text{-}4.2)$$

At present s is considered a constant. Since the terms on the left are functions of r only and those on the right are functions of z only, we can have equality only if each is equal to a constant independent of r and z. Let the constant be $-\lambda^2$. We then have two sets of ordinary differential equations:

$$rR_i'' + R_i' + \lambda^2 rR_i = 0 \qquad (1\text{-}4.3)$$

$$Z_i'' - \left(\frac{s^2}{c_i^2} + \lambda^2\right) Z_i = 0. \qquad (1\text{-}4.4)$$

Equation (1-4.3) is Bessel's differential equation of order zero. It has the solution $R_i = q_i J_0(\lambda r)$ which is finite at r = 0. The Bessel function $K_0(\lambda r)$ is also a solution of equation (1-4.3), but this function is infinite at r = 0. The function J_0 is the Bessel function of the first kind and of order zero. For our purposes $J_0(\lambda r)$ is best defined by the following integral representation (Dix [4]):

$$J_0(\lambda r) = \mathrm{Re}\left[\frac{2}{\pi}\int_0^{\pi/2} \exp(-i\lambda r \cos w)\, dw\right].* \qquad (1\text{-}4.5)$$

*Throughout this paper Im and Re denote respectively the imaginary and the real parts of a complex variable or function.

Equation (1-4.4) has solutions

$$Z_i = f_i \exp\left[\left(\frac{s^2}{c_i^2} + \lambda^2\right)^{1/2} z\right] + g_i \exp\left[-\left(\frac{s^2}{c_i^2} + \lambda^2\right)^{1/2} z\right].^{**} \quad (1\text{-}4.6)$$

The constants f_i, g_i, and q_i are to be determined from the initial conditions.

The "eigenvalue" λ can take on values from zero to infinity. Since equation (1-3.4) is linear, $\overline{P}_i(r,z,s)$ may be written as a continuous superposition of the product $R_i Z_i$ in the form of a Fourier-Bessel integral as follows:

$$\overline{P}_i(r,z,s) = \int_0^\infty \left\{a_i(\lambda) \exp\left[\left(\frac{s^2}{c_i^2} + \lambda^2\right)^{1/2} z\right]\right.$$

$$\left. + b_i(\lambda) \exp\left[-\left(\frac{s^2}{c_i^2} + \lambda^2\right)^{1/2} z\right]\right\} J_0(\lambda r)\lambda d\lambda \quad (1\text{-}4.7)$$

This is a general solution of the transformed wave equation for cylindrical symmetry. To apply this solution to the problem of bottom reflections, $a_i(\lambda)$ and $b_i(\lambda)$ must be determined so that the boundary conditions are satisfied.

The above general solution $\overline{P}_i$ does not satisfy the boundary condition 2(b) pertaining to the wave source. Hence, the solution is incomplete for the water layer. The expression $\overline{P}_1$ can be interpreted as that part of the wave which results from the bottom reflection but does not include the direct wave. The complete

---

**Throughout this paper the symbols $(\ )^{1/2}$ and $\sqrt{}$ denote, unless stated otherwise, the positive square root.

solution in the water is the sum of $\bar{P}_1$ above and the transform of the direct wave P_d which satisfies the wave equation (1-3.4) and the source condition 2(b).

An examination of the boundary condition 4(b) which refers to the signal strength at infinity shows that the coefficient $a_2(\lambda)$ must be zero for the wave in the bottom.

Abbreviating

$$\beta_i = (s^2 c_i^{-2} + \lambda^2)^{1/2} \qquad (1\text{-}4.8)$$

the complete solution in the water is

$$\bar{P}_1 = \int_0^\infty \left\{ a_1(\lambda) \exp(\beta_1 z) + b_1(\lambda) \exp(-\beta_1 z) \right\} J_0(\lambda r)\, \lambda d\lambda + \bar{P}_d, \qquad (1\text{-}4.9)$$

and the bottom solution is

$$\bar{P}_2 = \int_0^\infty \left\{ b_2(\lambda) \exp(-\beta_2 z) \right\} J_0(\lambda r)\, \lambda d\lambda. \qquad (1\text{-}4.10)$$

To obtain a solution of the bottom reflection problem an appropriate expression for the transformed source P_d must be introduced. A convenient form of the source is that of a spherical step wave because by means of the convolution integral, explained in Appendix D, solutions for any other wave form can be obtained. The subscript A will be used when reference to the step wave response is made. We choose the wave form P_A defined by

$$P_A = \frac{1}{R} H(\tau - R/c_1), \qquad (1\text{-}4.11)$$

where $H(\tau - R/c_1)$ is the Heaviside unit step function defined $H(\tau - R/c_1) = 0$ for $\tau \le R/c_1$ and $H(\tau - R/c_1) = 1$ for $\tau > R/c_1$. The expression R/c_1 is the arrival time of the direct wave. The Laplace transform of the step source is

$$\overline{P}_A = \frac{1}{sR} \exp(-sR/c_1) \quad . \tag{1-4.12}$$

Sommerfeld has derived the following identity for the transformed step wave:

$$\frac{1}{sR} \exp(-sR/c_1) = \frac{1}{s} \int_0^\infty J_0(\lambda r) \exp(-\beta_1 |z|) \frac{\lambda}{\beta_1} d\lambda \quad .* \tag{1-4.13}$$

This expression can be combined with the first integral of equation (1-4.9). The complete solution for the step wave and its reflection in the water is then

$$\overline{A}_1(r,z,s) = \int_0^\infty \left\{a_1(\lambda) \exp(\beta_1 z) + b_1(\lambda) \exp(-\beta_1 z)\right\} J_0(\lambda r) \lambda d\lambda$$
$$+ \frac{1}{s} \int_0^\infty J_0(\lambda r) \exp(-\beta_1 |z|) \frac{\lambda d\lambda}{\beta_1} \quad . \tag{1-4.14}$$

In the bottom the solution is

$$\overline{A}_2(r,z,s) = \int_0^\infty \left\{b_2(\lambda) \exp(-\beta_2 z)\right\} J_0(\lambda r) \lambda d\lambda \quad . \tag{1-4.15}$$

*A good derivation of this expansion is given by Ewing, Jardetzky, and Press [5].

The boundary condition 1(b) at the surface $z = -(D - h)$ requires that $\overline{P}_1(r,z,s) = 0$ at $z = -(D - h)$ and hence

$$a_1 \exp\left[-\beta_1 (D - h)\right] + b_1 \exp\left[\beta_1 (D - h)\right]$$
$$+ \frac{1}{\beta_1 s} \exp\left[-\beta_1 (D - h)\right] = 0. \qquad (1\text{-}4.16)$$

At the bottom-water interface $z = h$ the boundary conditions 3(b) require $\overline{P}_1 = \overline{P}_2$ and $(\partial\overline{P}_1/\partial z)/\rho_1 = (\partial\overline{P}_2/\partial z)/\rho_2$ and hence

$$a_1 \exp(\beta_1 h) + b_1 \exp(-\beta_1 h) + \frac{1}{\beta_1 s} \exp(-\beta_1 h) = b_2 \exp(-\beta_2 h) \qquad (1\text{-}4.17)$$

and

$$\frac{\beta_1}{\rho_1}\left[a_1 \exp(\beta_1 h) - b_1 \exp(-\beta_1 h) - \frac{1}{\beta_1 s} \exp(-\beta_1 h)\right] = \frac{-\beta_2}{\rho_2} b_2 \exp(-\beta_2 h). \qquad (1\text{-}4.18)$$

Solution of these three simultaneous equations yields

$$a_1 = \frac{1}{\beta_1 s}\left\{\frac{K\left[\exp(-2\beta_1 h) - \exp(-2\beta_1 D)\right]}{1 + K \exp(-2\beta_1 D)}\right\} \qquad (1\text{-}4.19)$$

$$b_1 = \frac{-\exp\left[-2\beta_1 (D - h)\right]}{\beta_1 s}\left\{\frac{1 + K \exp(-2\beta_1 h)}{1 + K \exp(-2\beta_1 D)}\right\} \qquad (1\text{-}4.20)$$

$$b_2 = \frac{(1 + K)}{\beta_1 s}\left\{\frac{\exp\left[h(\beta_2 - \beta_1)\right] - \exp\left[h(\beta_1 + \beta_2) - 2\beta_1 D\right]}{1 + K \exp(-2\beta_1 D)}\right\}, \qquad (1\text{-}4.21)$$

where the ratio of the density of the water to the density of the

bottom is abbreviated by

$$b = \rho_1/\rho_2 \tag{1-4.22}$$

and K is given by

$$K = \left[(\beta_1/\beta_2) - b\right]/\left[(\beta_1/\beta_2) + b\right] . \tag{1-4.23}$$

In later discussion (Appendix E) we will show that K can be interpreted as the plane wave reflection coefficient.

The solutions $\bar{A}_1(r,z,s)$ and $\bar{A}_2(r,z,s)$ can now be written:

In the water

$$\bar{A}_1 = \int_0^\infty \frac{1}{\beta_1 s} J_0(\lambda r)\lambda \left\{\frac{K\left[\exp(-2\beta_1 h) - \exp(-2\beta_1 D)\right]\exp(\beta_1 z)}{1 + K\exp(-2\beta_1 D)}\right.$$

$$- \frac{\exp\left[-2\beta_1(D - h)\right]\left[1 + K\exp(-2\beta_1 h)\right]\exp(-\beta_1 z)}{1 + K\exp(-2\beta_1 D)} \tag{1-4.24}$$

$$\left. + \exp(-\beta_1 |z|)\right\} d\lambda \quad ,$$

and in the bottom

$$\bar{A}_2 = \int_0^\infty \frac{1}{\beta_1 s} J_0(\lambda r)\lambda \left\{(1 + K)\left[\frac{\exp[h(\beta_2 - \beta_1)] - \exp[h(\beta_1 + \beta_2) - 2\beta_1 D]}{1 + K\exp(-2\beta_1 D)}\right]\right.$$

$$\left.\exp(-\beta_2 z)\right\} d\lambda \; . \tag{1-4.25}$$

These are the solutions of the transformed wave equation which satisfy the boundary conditions of our bottom reflection problem for a step wave source.

1-5. PEKERIS RAY EXPANSION

Since we are primarily concerned with the solution in the water we now restrict the discussion to $\bar{A}_1$ only. After some rearranging, equation (1-4.24) may be written

$$\bar{A}_1 = \int_0^\infty \frac{1}{\beta_1 s} J_0(\lambda r)\lambda \Big\{ \exp(-\beta_1 |z|) - \exp(\beta_1 z) \tag{1-5.1}$$

$$+ (\exp(\beta_1 z) - \exp[-\beta_1(2D - 2h + z)])$$

$$\cdot \left[\frac{1 + K \exp(-2\beta_1 h)}{1 + K \exp(-2\beta_1 D)}\right]\Big\} d\lambda \; .$$

Note that the term in brackets is the only term containing K. This expression can be expanded in a Taylor series about $K \exp(-2\beta_1 D) = 0$. Since $|K \exp(-2\beta_1 D)| < 1$, this series is uniformly convergent. We obtain the following expansion for $\bar{A}_1(r,z,s)$ in powers of K:

$$\bar{A}_1 = \frac{1}{s}\int_0^\infty J_0(\lambda r)\, \beta_1^{-1}\lambda \Big\{ K^\circ \big[\exp(-\beta_1|z|) - \exp(-\beta_1[2D - 2h + z])\big] \tag{1-5.2}$$

$$+ K^1\big[\exp(-\beta_1[2h - z]) - \exp(-\beta_1[2D - z]) + \exp(-\beta_1[4D - 2h + z])$$

$$- \exp(-\beta_1[2D + z])\big] + K^2\big[- \exp(-\beta_1[2D + 2h - z])$$

$$+ \exp(-\beta_1[4D - z] - \exp(-\beta_1[6D - 2h + z])$$

$$+ \exp(-\beta_1[4D + z])\big] + \ldots\Big\} d\lambda \; .$$

Since this series is uniformly convergent, it can be integrated term by term. The above "ray expansion" is credited to Bromwich [6] and is used by Pekeris [7]. The K^{0} terms contain the direct wave and the negative surface reflection. For each exponent n of K^{n}, $n \geq 1$, there are four terms which can be represented by four separate rays, denoted by m = 1, 2, 3, 4. These rays will be explained in more detail below. Physically, n denotes the number of times the ray has been reflected from the bottom.

1-6. IMAGE SOURCES

Each term in the above series expansion can be visualized as the signal from an image source located at $(0, {}_{n}d_{m})$, where ${}_{n}d_{m}$ is the height between the gauge and the image source (n, m) as shown in Figure 1-6.1. Each term in equation (1-5.2) can be written $(-1)^{n+m} K^{n} \exp(-\beta_{1}\ {}_{n}d_{m})$. To find the path of multiple reflections we can make use of the images and the "images of the images" with respect to the surface or the bottom as shown in Figure 1-6.1. We must remember that this method of images is based on the requirement that the angle of incidence equals the angle of reflection.

From equation (1-5.2) we can derive the following formulae for the four values of ${}_{n}d_{m}$ for a given $n \geq 1$:

$$ {}_{n}d_{1} = 2(n - 1)D + 2h - z \tag{1-6.1} $$

$$ {}_{n}d_{2} = 2n\ D - z \tag{1-6.2} $$

$$ {}_{n}d_{3} = 2(n + 1)D - 2h + z \tag{1-6.3} $$

$$ {}_{n}d_{4} = 2n\ D + z\ . \tag{1-6.4} $$

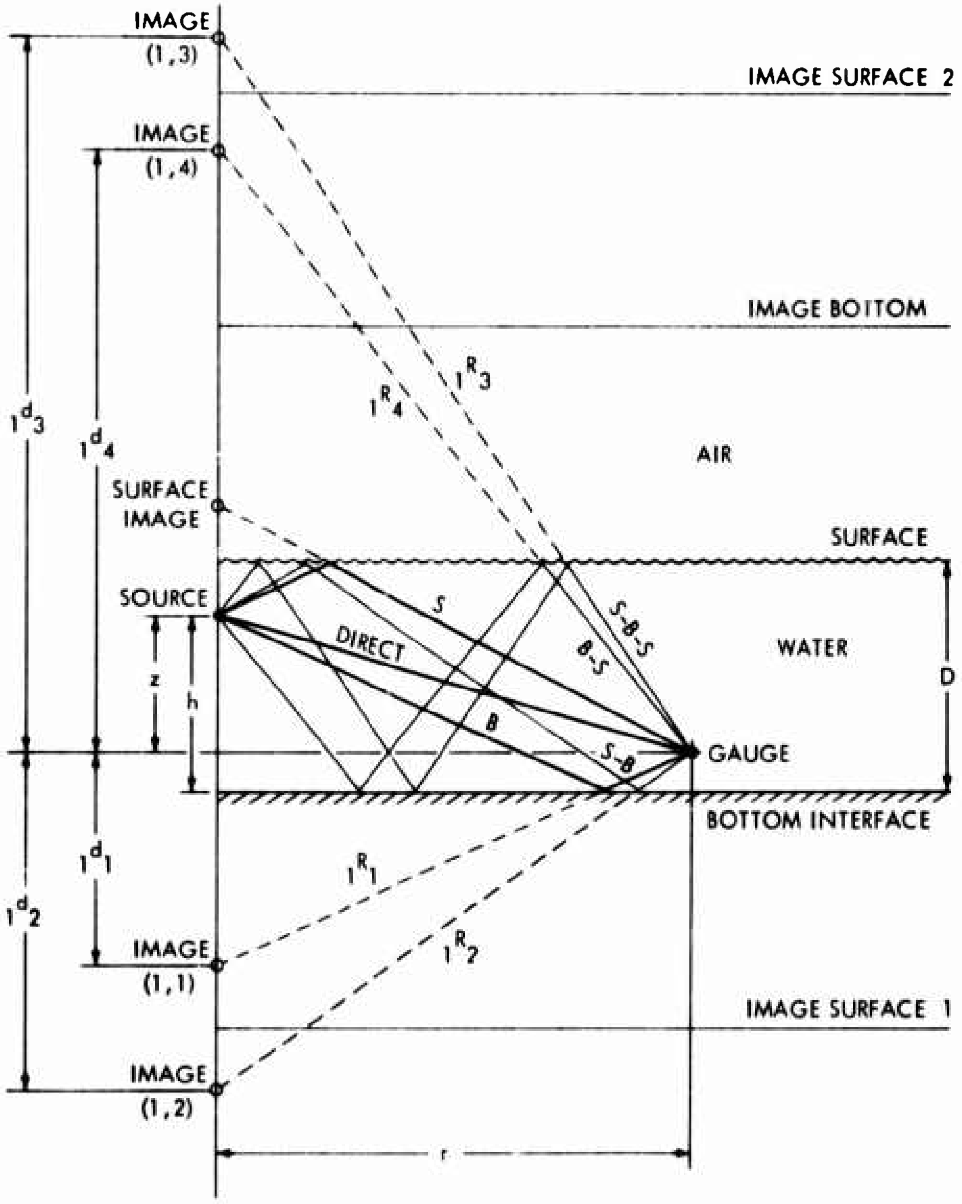

FIG. 1-6.1 REFLECTED RAYS AND IMAGES

The wave trains from the source to the gauge propagate along the rays shown as solid lines. To an observer at the gauge the trains seem to emerge from the various images. For instance, the wave train S-B appears to originate at IMAGE (1,2).

The location of IMAGE (1,2) can be determined by constructing the IMAGE SURFACE 1 which is the water surface mirrored in the bottom interface. IMAGE (1,2) is then IMAGE (1,1) mirrored in the IMAGE SURFACE 1, and IMAGE (1,1) is in turn the source mirrored in the bottom interface.

For n = 0 the surface reflection and the direct wave are denoted respectively by m = 1 and m = 2. The equations above then reduce to the following:

$$_{0}d_{1} = 2D - 2h + z \tag{1-6.5}$$

$$_{0}d_{2} = |z| \ . \tag{1-6.6}$$

The significance of the four values of m becomes evident if we use the following notation (Rosenbaum [8]). Let B represent a reflection from the bottom and S represent a reflection from the surface. Then, for n = 1, we have the following reflections for m = 1, 2, 3, and 4 respectively: B, S - B, S - B - S, and B - S. Similarly, for n = 2, there are the following reflections for m = 1, 2, 3, and 4 respectively: B- S - B, S - B - S - B, S - B - S - B - S, and B - S - B - S.

It is interesting to note that for a given value of $n \geq 1$ the four values of m represent the combinations of alternately S and B which contain the letter B repeated n times.

1-7. INTRODUCTION OF AN INTEGRAL REPRESENTATION FOR THE BESSEL FUNCTION $J_0(\lambda r)$

Each term in equation (1-5.2) is the Laplace-transformed step wave pressure response resulting from image source (n,m). For $n \geq 1$ we denote these transformed pressures by ${}_n\overline{P}_m$. The total transformed pressure in the water is then

$$\overline{A}_1(r,z,s) = {}_o\overline{P}_1 + {}_o\overline{P}_2 + \sum_{m=1,2,\ldots}^{4} \sum_{n=1,2,\ldots}^{\infty} {}_n\overline{P}_m , \qquad (1\text{-}7.1)$$

where ${}_o\overline{P}_1$ and ${}_o\overline{P}_2$ represent respectively the surface reflection and the direct wave terms. The general expression for ${}_n\overline{P}_m$ is then obtained from equation (1-5.2):

$${}_n\overline{P}_m = (-1)^{n+m} \frac{1}{s} \int_0^{\infty} J_0(\lambda r)\ \beta_1{}^{-1} \lambda K^n \exp\left(-\beta_1\, {}_n d_m\right) d\lambda . \qquad (1\text{-}7.2)$$

The Bessel function $J_0(\lambda r)$ may be written as the following integral:

$$J_0(\lambda r) = \mathrm{Re}\left[\frac{2}{\pi} \int_0^{\pi/2} \exp\left(-i\lambda r \cos w\right) dw\right] . \qquad (1\text{-}7.3)$$

Substitution of this integral for $J_0(\lambda r)$ in equation (1-7.2) yields

$${}_n\overline{P}_m = (-1)^{n+m} \frac{1}{s} \int_0^{\infty} \left\{\mathrm{Re}\left[\frac{2}{\pi} \int_0^{\pi/2} \exp(-i\lambda r \cos w) dw\right]\right.$$

$$\left.\beta_1{}^{-1} \lambda K^n \exp\left(-\beta_1\, {}_n d_m\right)\right\} d\lambda . \qquad (1\text{-}7.4)$$

Making the change of variable $su = \lambda$ with u real and interchanging the order of integration, ${}_n\bar{P}_m$ may be written

$$ {}_n\bar{P}_m = (-1)^{n+m} \,\mathrm{Re} \left\{ \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\infty} u\alpha_1^{-1} K^n \right. $$

$$ \left. \exp\left[-s(\alpha_1 \, {}_nd_m + iur \cos w)\right] du \, dw \right\} , \qquad (1\text{-}7.5) $$

where $\alpha_1 = (u^2 + c_1^{-2})^{1/2}$ (1-7.6)

$$ \alpha_2 = (u^2 + c_2^{-2})^{1/2} \qquad (1\text{-}7.7) $$

$$ K = (\alpha_1 - b\alpha_2)/(\alpha_1 + b\alpha_2) \; . \qquad (1\text{-}7.8) $$

Although u will be later treated as a complex variable, u is real in the above integrals because both s and λ are real. This means the integration above is performed along the real axis of the complex u - plane. To carry out the integration the path will later be changed to pass over complex u.

1-8. THE CHANGE OF VARIABLES (u,τ)

In equation (1-7.5) the term in brackets suggests the change in variables at constant w defined by

$$ \tau = \alpha_1 \, {}_nd_m + iur \cos w \; . \qquad (1\text{-}8.1) $$

The integral of equation (1-7.5) then becomes

$$ {}_n\overline{P}_m = (-1)^{n+m} \,\mathrm{Re}\left\{\frac{2}{\pi} \cdot \int_0^{\pi/2} \left[\int_{c_1^{-1}\,{}_nd_m}^{\infty} u\alpha_1^{-1} K^n \left(\frac{\partial u}{\partial \tau}\right)_w \exp(-s\tau)\, d\tau \right] dw \right\} \quad . \tag{1-8.2} $$

In the second integral the formal change of variables (u,τ) leads to an integration path in the first quadrant of the complex τ - plane. However, it is shown in Appendix C that the path of integration can be shifted back to the real axis of the τ - plane. That is, the integration is performed for real values of τ. This variable τ, defined by equation (1-8.1), is shown in Appendix F to be equivalent to the time τ when ${}_nP_m$ is evaluated for $n = 0$, that is, for the direct wave and surface reflection.

1-9. THE INVERSE LAPLACE TRANSFORM OF ${}_n\overline{P}_m$

A change of the order of integration in the above equation is permissible and yields

$$ {}_n\overline{P}_m = (-1)^{n+m} \,\mathrm{Re}\left\{\frac{2}{\pi} \int_{c_1^{-1}\,{}_nd_m}^{\infty} \left[\int_0^{\pi/2} u\alpha_1^{-1} K^n \left(\frac{\partial u}{\partial \tau}\right)_w dw \right] \exp(-s\tau)\, d\tau \right\} \quad . \tag{1-9.1} $$

This integral may be interpreted as the Laplace transform of a function with the following properties:

$$ {}_nP_m = 0 \;, \quad \tau < c_1^{-1}\,{}_nd_m $$

$$ {}_nP_m = (-1)^{n+m} \,\mathrm{Re}\left\{\frac{2}{\pi} \int_0^{\pi/2} u\alpha_1^{-1} K^n \left(\frac{\partial u}{\partial \tau}\right)_w dw \right\} \;, \quad \tau \geq c_1^{-1}\,{}_nd_m . \tag{1-9.2} $$

It is shown in Appendix B that the pressure ${}_nP_m$ is also zero for $\tau < {}_n\delta_m$ where $c_1^{-1}\,{}_nd_m \leq {}_n\delta_m \leq c_1^{-1}\,{}_nR_m$. The magnitude ${}_n\delta_m$ is the earliest arrival time of waves (n,m) reaching the receiver and is discussed in detail, along with other characteristic times of the pressure response, in Appendix B. Therefore, the above expression for ${}_nP_m$ is the desired solution of our problem. The change of variable (u,τ) and the interpretation of equation (1-9.1) as a Laplace transform is the vital step of Cagniard's analysis which makes trivial the inversion of the Laplace transform ${}_n\overline{P}_m$ and eliminates the integration to infinity.

In the following paragraph a further change of variable is made which simplifies the integral (1-9.2) and its interpretation.

1-10. THE CHANGE OF VARIABLE (w,u)

A change of variable from w to u at constant τ yields for $\tau \geq {}_n\delta_m$

$$ {}_nP_m = (-1)^{n+m}\,\mathrm{Re}\left\{\frac{2}{\pi}\int_{u_1}^{u_2} u\alpha_1^{-1}\,K^n\left(\frac{\partial u}{\partial \tau}\right)_w\left(\frac{\partial w}{\partial u}\right)_\tau du\right\}, \tag{1-10.1} $$

where

$$ u_1 = {}_nR_m^{-2}\left[-i\,\tau r + {}_nd_m\,(\tau^2 - c_1^{-2}\,{}_nR_m^2)^{1/2}\right], \tag{1-10.2} $$

$$ u_2 = {}_nd_m^{-1}\,(\tau^2 - c_1^{-2}\,{}_nd_m^2)^{1/2}, \tag{1-10.3} $$

and

$$ {}_nR_m = (r^2 + {}_nd_m^2)^{1/2}. $$

From equation (1-8.1) we can write

$$ \left(\frac{\partial w}{\partial u}\right)_\tau = \left(ir\cos w + \frac{u\,{}_nd_m}{\alpha_1}\right)/(iur\sin w) = \left[\left(\frac{\partial u}{\partial \tau}\right)_w (iur\sin w)\right]^{-1}. \tag{1-10.4} $$

From equation (1-8.1) we also obtain

$$\sin w = (1 - \cos^2 w)^{1/2} = \left[u^2 r^2 + (\tau - \alpha_1 \; {}_n d_m)^2\right]^{1/2} / ur \; . \qquad (1\text{-}10.5)$$

Hence for $\tau \geq {}_n\delta_m$

$$ {}_nP_m = (-1)^{n+m} \operatorname{Re} \left\{ \frac{2}{\pi i} \int_{u_1}^{u_2} u\alpha_1^{-1} K^n \right. \qquad (1\text{-}10.6)$$

$$\left. \left[u^2 r^2 + (\tau - {}_n d_m \, \alpha_1)^2\right]^{-1/2} du \right\} .$$

1-11. ELIMINATION OF THE OPERATOR Re

The above expression for ${}_nP_m$ can be further simplified by eliminating the operator Re which can be accomplished as follows: Any complex quantity W has a real part given by

$$\operatorname{Re}(W) = (W + \widetilde{W})/2 \; , \qquad (1\text{-}11.1)$$

where the tilde denotes the complex conjugate. Let ${}_nP_m$ be denoted by

$$ {}_nP_m = \operatorname{Re} \int_{u_1}^{u_2} f(u) \, du. \qquad (1\text{-}11.2)$$

We can then write

$$ {}_nP_m = \frac{1}{2} \int_{u_1}^{u_2} f(u) \, du + \frac{1}{2} \widetilde{\int_{u_1}^{u_2} f(u) \, du} \; , \qquad (1\text{-}11.3)$$

where the tilde over the integral denotes the complex conjugate. In general, the complex conjugate of an integral can be written as follows:

$$\widetilde{\int_{u_1}^{u_2} f(u)\ du} = \int_{\tilde{u}_1}^{\tilde{u}_2} \widetilde{f(u)}\ d\tilde{u}\ . \qquad (1\text{-}11.4)$$

But for the present case each expression in f(u) is composed of sums, products, quotients, and powers of polynomial functions of u and u^2. Each component function g(u) of f(u) satisfies the relations

$$g(u) = w + iv \qquad (1\text{-}11.5)$$

$$g(\tilde{u}) = w - iv\ , \qquad (1\text{-}11.6)$$

where w and v denote respectively the real and imaginary parts of g(u). But f(u) also has the constant factor $1/i = -i$ which does not satisfy equations (1-11.5) and (1-11.6). Hence we write f(u) in the form

$$f(u) = -i(W + iV) = -iW + V\ , \qquad (1\text{-}11.7)$$

where W and V denote respectively the real and imaginary parts of i f(u). The sum W + iV satisfies equations (1-11.5) and (1-11.6) and we obtain

$$f(\tilde{u}) = -i(W - iV) \qquad (1\text{-}11.8)$$

and

$$\widetilde{f(u)} = -i\widetilde{(W - iV)} = iW + V. \qquad (1\text{-}11.9)$$

Equations (1-11.8) and (1-11.9) yield

$$f(\tilde{u}) = -\widetilde{f(u)} . \qquad (1\text{-}11.10)$$

We can then write

$$_nP_m = \frac{1}{2}\int_{u_1}^{u_2} f(u)\,du - \frac{1}{2}\int_{\tilde{u}_1}^{\tilde{u}_2} f(\tilde{u})\,d\tilde{u} . \qquad (1\text{-}11.11)$$

Interchanging the limits of integration in the second integral and noting that $\tilde{u}_2 = u_2$, we have

$$_nP_m = \frac{1}{2}\int_{u_1}^{u_2} f(u)\,du + \frac{1}{2}\int_{u_2}^{\tilde{u}_1} f(\tilde{u})\,d\tilde{u}. \qquad (1\text{-}11.12)$$

These integrals can be combined into one

$$_nP_m = \frac{1}{2}\int_{u_1}^{\tilde{u}_1} f(u)\,du , \qquad (1\text{-}11.13)$$

since the integration variable $\tilde{u}$ in the second integral of (1-11.12) can be replaced by u. Substituting the expanded expression for f(u) we have

$$_nP_m = 0 \qquad \tau < {}_n\delta_m$$

$$_nP_m = (-1)^{n+m}\frac{1}{\pi i}\int_{u_1}^{\tilde{u}_1} u\,\alpha_1^{-1}\,K^n\left[u^2 r^2 + (\tau - {}_nd_m\,\alpha_1)^2\right]^{-1/2} du , \quad \tau \geq {}_n\delta_m .$$
(1-11.14)

where $K = (\alpha_1 - b\alpha_2)/(\alpha_1 + b\alpha_2)$. Setting

$$\gamma = \left[u^2 r^2 + (\tau - {}_n d_m \, \alpha_1)^2\right]^{1/2} , \tag{1-11.15}$$

the step wave response ${}_n P_m$ can be written

$$\begin{aligned} {}_n P_m &= 0 & \tau &< {}_n \delta_m \\ {}_n P_m &= (-1)^{n+m} \frac{1}{\pi i} \int_{u_1}^{\bar{u}_1} u \frac{K^n}{\alpha_1 \gamma} \, du & \tau &\geq {}_n \delta_m \, . \end{aligned} \tag{1-11.16}$$

The above expression for ${}_n P_m$ is the solution of the wave equation for the reflection of a step wave

$$P_A = H(\tau - c_1^{-1} R)/R \tag{1-11.17}$$

from a liquid bottom. The notation (n,m), introduced in paragraph 1-5, identifies the number of the reflections from the bottom, n, and a particular path m. Solutions for an arbitrary pressure pulse can be obtained from the step wave solution by means of the convolution integral explained in Appendix D.

The solution (1-11.14) or (1-11.16) can be evaluated in two ways: (a) by analytically separating out the real part as done by Rosenbaum and described in Chapter II or (b) by a numerical separation of the real part as proposed by Eichler and Rattayya [9] and explained in Chapter V.

The integration must be made numerically in either case, and the characteristic times explained in Appendix B must be taken into

account. In the first case only real variables are involved in the final computation. The second case requires a computer program which uses complex variables such as provided for FORTRAN IV. The first method takes less computer time although the machine program is more complicated. On the other hand it will be seen in Appendix A paragraph A-13 that the expressions for multiple reflections from rigid bottoms become so complicated that Eichler and Rattayya's proposal becomes almost essential.

1-12 CONCLUDING REMARKS

In the derivation of the solution ${}_nP_m$, Eq. (1-11.14), a number of mathematical details have been bypassed for the sake of a straightforward presentation. These details are described in Appendices B, C, D... and the reader interested in a complete picture is urged to read these at this point.

Many of the mathematical operations use complex variables. For the sake of the reader unfamiliar with the theory of complex variables a few salient points of the theory are summarized in Appendix A.

CHAPTER II

THE LIQUID BOTTOM SOLUTION IN THE FORM OF ROSENBAUM

In this chapter we derive the equations given by Rosenbaum [2] for the step wave pressure response ${}_1P_m$ when the bottom is a fluid. General equations for n>1 are not derived in this paper. The general equations for a liquid bottom only are given in Rosenbaum's paper, but since they are complicated and since corresponding equations for a rigid bottom would be exceedingly complicated, it is suggested that ${}_nP_m$ for higher ordered reflections be calculated by the method explained in Chapter V.

We begin with equation (1-11.14) which corresponds to equation (15) of Rosenbaum's paper except for the factor $(-1)^{n+m}$ which he neglects and a scale factor G which he includes but we set equal to unity. To simplify the notation we will usually drop the subscript n (except in ${}_nP_m$), and the variables R_m, d_m, etc. will be understood to mean ${}_nR_m$, ${}_nd_m$, etc. Also δ will be taken to mean ${}_n\delta_m$. Even though it may be confusing, we will continue to use Rosenbaum's notation and let $\tau_m = \tau/R_m$ and $\delta_m = \delta/R_m$.

2-1. PRECURSOR INTEGRAL (Case $c_2 > c_1$)

When $\tau < c_1^{-1} R_m$ the limits of integration are imaginary since the square root $(\tau - c_1^{-1} R_m)^{1/2}$ is imaginary. The branch cuts for the square roots α_1, α_2, and γ are made respectively along the imaginary axis from $-ic_1^{-1}$ to ic_1^{-1}, $-ic_2^{-1}$ to ic_2^{-1}, and u_1 to $\tilde{u}_1$ as explained in Appendix A. The integration is performed on the

Riemann sheet on which the radicals have positive real parts for positive, real u. The integration is then carried out along the imaginary axis from u_1 to $\tilde{u}_1$ as explained in Appendix B and diagrammed in Figure 2-1.1 below. It is also shown in Appendix B

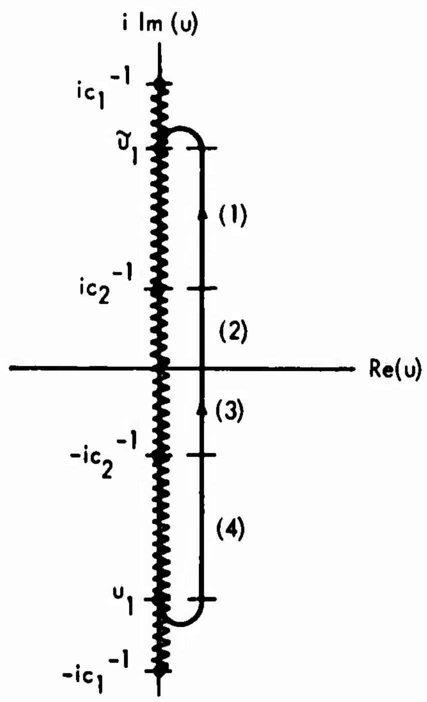

FIG. 2-1.1 INTEGRATION PATH FOR LIQUID BOTTOM PRECURSOR ($c_2 > c_1$ AND $\delta \leq \tau < c_1^{-1}R_m$)

that paths (2) and (3) of this figure from $-ic_2^{-1}$ to ic_2^{-1} make no net contribution to the integral since the square roots α_1 and α_2 are real along these paths. The pressure ${}_nP_m$ is then the sum of contributions from paths (1) and (4) which approach the imaginary axis from the right in the limit. The end paths which join the vertical paths (1) and (4) to the integration limits u_1 and $\tilde{u}_1$ approach zero length, as the vertical paths approach the imaginary axis, and do not contribute to the integration.

Let ${}_nP'_m$ denote the integrand of ${}_nP_m$. Then

$$ {}_nP_m = \int_{u_1}^{\tilde{u}_1} {}_nP'_m \, du = (1)\int_{ic_2^{-1}}^{\tilde{u}_1} {}_nP'_m \, du + (4)\int_{u_1}^{-ic_2^{-1}} {}_nP'_m \, du \, . \qquad (2\text{-}1.1) $$

Since the integrand is a function of u^2 only, we may replace -u by u. Then after interchanging the limits of integration on path (1),

$$ {}_nP_m = -(1)\int_{\tilde{u}_1}^{ic_2^{-1}} {}_nP'_m \, du + (4)\int_{\tilde{u}_1}^{ic_2^{-1}} {}_nP'_m \, du \, . \qquad (2\text{-}1.2) $$

The labels indicating the paths are retained so that the proper signs can be chosen for the square roots.

The square roots α_1 and γ are real on paths (1) and (4), but α_2 is imaginary. The imaginary part of α_2 is positive on path (1) and negative on path (4). Since α_2 is contained in K, we must determine the real and imaginary parts of K to evaluate the real part of ${}_nP_m$.

Since the algebra becomes increasingly complicated, from this point the derivation will be specialized to the case n = 1. As will be explained in Chapter V, pressure histories for n > 1 can be more readily obtained using complex arithmetic to evaluate equation (1-11.14). It must be remembered that the n = 1 case represents rays which have been reflected once from the bottom and that this case, particularly for m = 1, makes the most significant contributions to the pressure time history.

The liquid bottom reflection coefficient K when α_1 is real and positive, but α_2 is imaginary and positive, can be written using real square roots as follows:

$$K = \frac{\alpha_1 - ib\,[-(u^2 + c_2{}^{-2})]^{1/2}}{\alpha_1 + ib\,[-(u^2 + c_2{}^{-2})]^{1/2}} \; . \qquad (2\text{-}1.3)$$

Separating real and imaginary parts of K, one obtains

$$\mathrm{Im}(K) = \frac{-2b\,\alpha_1 [-(u^2 + c_2{}^{-2})]^{1/2}}{\alpha_1{}^2 - b^2(u^2 + c_2{}^{-2})} \quad \text{and}$$

$$\mathrm{Re}(K) = \frac{\alpha_1{}^2 + b^2\,(u^2 + c_2{}^{-2})}{\alpha_1{}^2 - b^2\,(u^2 + c_2{}^{-2})} \; . \qquad (2\text{-}1.4)$$

The real part of ${}_1P_m$ comes from Im(K). The imaginary part of ${}_nP_m$ is identically zero since ${}_nP_m$ is the real part of a complex integral. (See equation (1-10.6).) Since $\mathrm{Im}(\alpha_2)$ is positive on path (1), but is negative on path (4), Im(K) has opposite signs on the two paths. All other expressions in the integrand ${}_1P'_m$ do not change in sign from path (1) to path (4). Hence ${}_nP'_m$ also has opposite signs on the two paths, and the integrals of equation (2-1.2) can be combined into one integral. Equations (2-1.2) and (1-11.16) then yield

$${}_1P_m = (-1)^m \frac{2}{\pi} \int_{\tilde{u}_1}^{ic_2{}^{-1}} \frac{u\,\mathrm{Im}(K)}{\alpha_1\,\gamma}\,du \; . \qquad (2\text{-}1.5)$$

Making the change of variable u = ix and substituting Im(K) from above, ${}_1P_m$ becomes

$${}_1P_m = (-1)^m \frac{4b}{\pi} \int_{\mathrm{Im}(\tilde{u}_1)}^{c_2{}^{-1}} \frac{(x^2 - c_2{}^{-2})^{1/2}\,x\,dx}{\gamma[\alpha_1{}^2 + b^2(x^2 - c_2{}^{-2})]} \, , \qquad (2\text{-}1.6)$$

where γ and α_1 are expressed as functions of x and

$$\mathrm{Im}(\tilde{u}_1) = R_m^{-2}\left[\tau r - d_m(c_1^{-2} R_m^{2} - \tau^2)^{1/2}\right] .$$

After Rosenbaum, we introduce the following notation:

$$\cos \ell_m = \frac{d_m}{R_m},\ \sin \ell_m = \frac{r}{R_m},\ \omega = (c_1^{-2} - x^2)^{1/2},$$

$$\sigma = (c_1^{-2} - c_2^{-2})^{1/2},\ \tau_m = \frac{\tau}{R_m},\ \delta_m = \frac{\delta}{R_m} = c_2^{-1} \sin \ell_m + \sigma \cos \ell_m,$$

$$M_m = \tau_m \cos \ell_m + (c_1^{-2} - \tau_m^{2})^{1/2} \sin \ell_m,$$

$$N_m = \tau_m \cos \ell_m - (c_1^{-2} - \tau_m^{2})^{1/2} \sin \ell_m, \text{ and}$$

$$\omega = (\sigma + M_m)/2 + [(\sigma - M_m)/2] \sin \frac{\pi\psi}{2} .$$

Making the substitutions ω, τ_m, σ, $\sin \ell_m$, and $\cos \ell_m$ in equation (2-1.6), we obtain

$$ {}_1P_m = (-1)^{m+1} \frac{4b}{\pi R_m} \int_{\omega_1}^{\omega_2} \left\{\omega(\sigma^2 - \omega^2)^{1/2} d\omega\right\} \Big/ \left\{\left[\omega^2 + b^2(\sigma^2 - \omega^2)\right]\right.$$

$$\left.\left[(\omega^2 - c_1^{-2}) \sin^2 \ell_m + \tau_m^{2} - 2\tau_m \omega \cos \ell_m + \omega^2 \cos^2 \ell_m\right]^{1/2}\right\} . \tag{2-1.7}$$

Substituting N_m, M_m, and $d\omega = [\pi(\sigma - M_m)/4] \cos (\pi\psi/2)\, d\psi$, we find

$$ {}_1P_m = (-1)^{m+1} \frac{b(\sigma - M_m)}{R_m} \int_{\psi_{m1}}^{\psi_{m2}} \frac{\omega(\sigma^2 - \omega^2)^{1/2} \cos \frac{\pi\psi}{2}\, d\psi}{[\omega^2 + b^2(\sigma^2 - \omega^2)][\omega^2 + N_m M_m - \omega(N_m + M_m)]^{1/2}} . \tag{2-1.8}$$

where we have used $N_m M_m = \tau_m^2 - c_1^{-2} \sin^2 \ell_m$ and $N_m + M_m = 2\tau_m \cos \ell_m$.

The limits of integration must now be determined. At $x = c_2^{-1}$, we have $\omega = (c_1^{-2} - c_2^{-2})^{1/2} = \sigma$. Since $\omega = (\sigma + M_m)/2 + [(\sigma - M_m)/2] \sin \pi\psi/2$, we get that $\sin \pi\psi/2 = 1$ at $\omega = \sigma$ and hence $\psi_{m2} = 1$. At $x = \mathrm{Im}(\tilde{u}_1)$ it is easily shown that $\omega = M_m$, and consequently that $\psi_{m1} = -1$.

Returning to equation (2-1.8), it can be shown using trignometric identities that

$$\cos \frac{\pi\psi}{2} = \frac{(\omega - M_m)^{1/2}}{(\sigma - \omega)^{1/2}} \left[1 - \sin \frac{\pi\psi}{2}\right] . \tag{2-1.9}$$

Thus the change of variable to ψ was made so that the factor $(\omega - M_m)$ of γ could be eliminated, removing the singularity of γ at $u = \tilde{u}_1$ (or $\omega = M_m$ and $\psi_{m1} = -1$). Substituting the above expression for $\cos \pi\psi/2$ and canceling $(\sigma - \omega)^{1/2}(\omega - M_m)^{1/2}$ from numerator and denominator, we obtain

$$ {}_1P_m = 0 \qquad (\tau_m < \delta_m)$$

$$ {}_1P_m = (-1)^{m+1} \frac{b(\sigma - M_m)}{R_m} \int_{-1}^{1} \frac{\omega(\sigma+\omega)^{1/2}[1-\sin \frac{\pi\psi}{2}]\, d\psi}{[(1-b^2)\omega^2+\sigma^2 b^2](\omega - N_m)^{1/2}}$$

$$(\delta_m \le \tau_m \le c_1^{-1}) \tag{2-1.10}$$

The above equation is Rosenbaum's form of ${}_1P_m(\tau_m)$ for $\tau_m < c_1^{-1}$ (except for the factor $(-1)^{m+1}$ and the scale factor G). When $\tau_m < \delta_m$, we have ${}_1P_m(\tau_m) = 0$ since the precursor, the first arrival, has not yet arrived. Replacing the terms in equation (2-1.10) resulting from Im(K) by corresponding terms from Im(K^n) one can

obtain Rosenbaum's results for higher ordered reflections since $Im(K^n)$ has the same sign dependence on α_2 as does $Im(K)$.

2-2 MAIN WAVE INTEGRAL (Case $c_2 > c_1$)

The main wave integration is quite similar to the precursor integral except that u_1 and $\tilde{u}_1$ are now complex. The branch cuts for α_1 and α_2 are the same as before, but the branch cut for γ is made from u_1 in the fourth quadrant to $\tilde{u}_1$ in the first quadrant. The square root γ now has a positive real part left of the cut and a negative real part right of the cut. The initial integration path is shown in Figure 2-2.1 as the line $u_1 A \tilde{u}_1$.

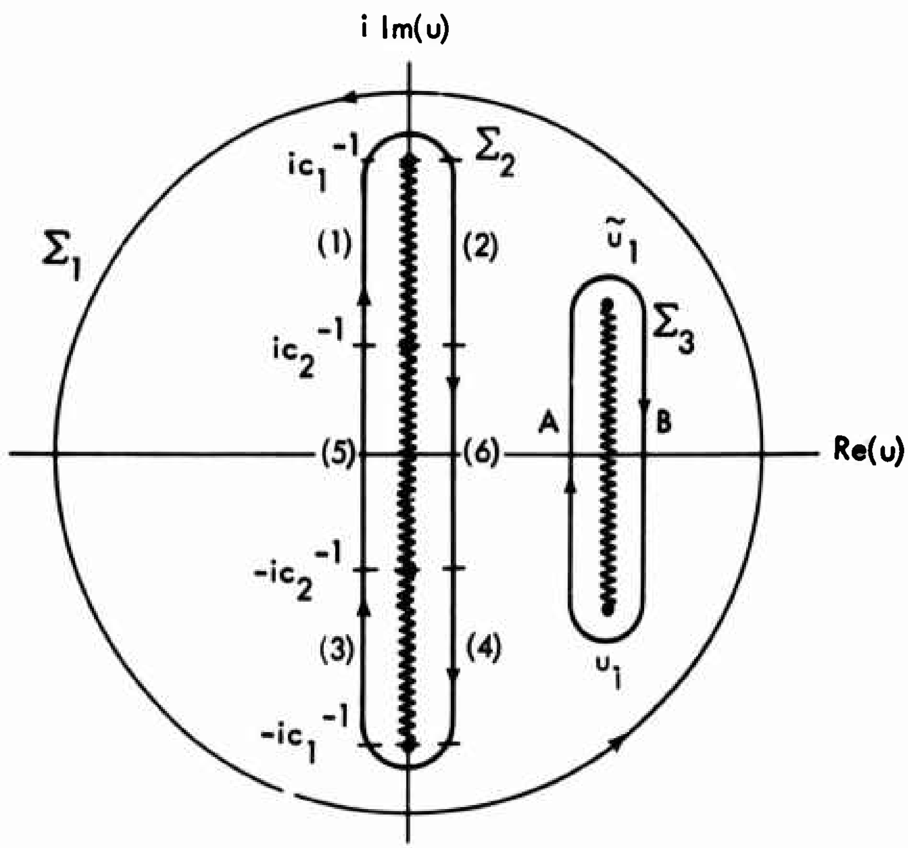

FIG. 2-2.1 PATHS OF INTEGRATION IN THE u-PLANE FOR $\tau > c_1^{-1} R_m$ (LIQUID BOTTOM $c_2 > c_1$)

Since $\mathrm{Re}(\gamma)$ is positive on $u_1 A \tilde{u}_1$, but negative on $u_1 B \tilde{u}_1$, and since α_1 and α_2 have the same signs on both paths, we can write

$$ {}_nP_m = \frac{1}{2} \oint_{\Sigma_0} {}_nP'_m \, du \ . \tag{2-2.1}$$

Cauchy's theorem allows us to write

$$ {}_nP_m = \frac{1}{2} \oint_{\Sigma_0} {}_nP'_m \, du = -\frac{1}{2} \left\{ \oint_{\Sigma_1} {}_nP'_m \, du + \oint_{\Sigma_2} {}_nP'_m \, du \right\} . \tag{2-2.2}$$

As the radius of Σ_1 tends to infinity, ${}_nP'_m$ has a pole at infinity which in Appendix A is shown to have the residue

$$ R_\infty = \frac{-(-1)^{n+m}}{i\pi R_m} \left(\frac{1-b}{1+b} \right)^n \ . \tag{2-2.3}$$

Since the contour Σ_1 is in the positive (counterclockwise) direction, we apply the residue theorem to obtain

$$ {}_nP_m = \frac{(-1)^{n+m}}{R_m} \left(\frac{1-b}{1+b} \right)^n - \frac{1}{2} \oint_{\Sigma_2} {}_nP'_m \, du \ . \tag{2-2.4}$$

For more details of the procedure used to obtain the above equation see Appendix A.

Denote the integral on Σ_2 by I_2. Then, dividing Σ_2 into the paths (1), (2), (3), (4), (5), and (6) shown in Figure 2-2.1, the integrals along (5) and (6) between $-ic_2^{-1}$ and ic_2^{-1} add to zero as in the precursor leaving

$$ I_2 = (1)\int_{ic_2^{-1}}^{ic_1^{-1}} {}_nP'_m \, du + (2)\int_{ic_1^{-1}}^{ic_2^{-1}} {}_nP'_m \, du \tag{2-2.5}$$

$$ + (3)\int_{-ic_1^{-1}}^{-ic_2^{-1}} {}_nP'_m \, du + (4)\int_{-ic_2^{-1}}^{-ic_1^{-1}} {}_nP'_m \, du \ .$$

Replacing -u by u on paths (3) and (4) and interchanging the limits of integration on paths (1) and (4), we obtain

$$I_2 = -(1)\int_{ic_1^{-1}}^{ic_2^{-1}} {}_nP_m' \, du + (2)\int_{ic_1^{-1}}^{ic_2^{-1}} {}_nP_m' \, du \tag{2-2.6}$$

$$+ (3)\int_{ic_1^{-1}}^{ic_2^{-1}} {}_nP_m' \, du - (4)\int_{ic_1^{-1}}^{ic_2^{-1}} {}_nP_m' \, du \, .$$

The labels indicating the original paths of integration are retained so that the proper signs can be chosen for the complex square roots. Again we restrict the derivation to the case n = 1.

On paths (2) and (4) α_1 is real and positive, but on paths (1) and (3) it is real and negative. Let γ_1 denote the value of γ for positive α_1, let γ_2 denote the value of γ for negative α_1. As in the precursor the real part of ${}_nP_m$ comes from Im(K), and the integrals on paths (1) and (3) can be combined since Im(K), given by equation (2-1.4), is positive on (1) and negative on (3) because of the imaginary square root α_2. Similarly the integral on paths (2) and (4) may be added. On all four paths γ_1 and γ_2 are real and positive. The four integrals of equation (2-2.6) can then be combined into one integral and I_2 may be written

$$I_2 = (-1)^m \frac{4b}{\pi} \int_{ic_1^{-1}}^{ic_2^{-1}} \frac{u[-(u^2+c_2^{-2})]^{1/2}\alpha_1}{\alpha_1[\alpha_1^2-b^2(u^2+c_2^{-2})]} \left\{\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right\} du \, . \tag{2-2.7}$$

Making the substitution u = ix and canceling the α_1's, we have

$$I_2 = (-1)^{m+1} \frac{4b}{\pi} \int_{c_1^{-1}}^{c_2^{-1}} \frac{x(x^2-c_2^{-2})^{1/2}}{[\alpha_1^2+b^2(x^2-c_2^{-2})]} \left\{\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right\} dx \, . \tag{2-2.8}$$

The following Rosenbaum substitutions are used: $\tau_m = \tau/R_m$, $\omega = (c_1^{-2} - x^2)^{1/2}$, $\sigma = (c_1^{-2} - c_2^{-2})^{1/2}$, $\cos \ell_m = d_m/R_m$, $\sin \ell_m = r/R_m$, $K_m = \tau_m \cos \ell_m$, and $L_m = (\tau_m^2 - c_1^{-2}) \sin^2 \ell_m$. At the lower limit, $x = c_1^{-1}$, ω becomes $\omega_1 = 0$, and at the upper limit, $x = c_2^{-1}$, ω becomes $\omega_2 = \sigma$. Thus the integral I_2 on Σ_2 can be written

$$I_2 = (-1)^m \frac{4b}{\pi R_m} \int_0^\sigma \frac{\omega(\sigma^2 - \omega^2)^{1/2}}{[(1 - b^2)\omega^2 + \sigma^2 b^2]} \tag{2-2.9}$$

$$\left\{[(\omega - K_m)^2 + L_m]^{-1/2} - [(\omega + K_m)^2 + L_m]^{-1/2}\right\} d\omega.$$

Substituting I_2 above into equation (2-2.4), we get Rosenbaum's result for $\tau > c_1^{-1} R_m$ and $c_2 > c_1$:

$${}_1P_m = \frac{(-1)^{m+1}}{R_m} \frac{1 - b}{1 + b} \tag{2-2.10}$$

$$+ (-1)^{m+1} \frac{2b}{\pi R_m} \int_0^\sigma \frac{\omega(\sigma^2 - \omega^2)^{1/2}}{[(1 - b^2)\omega^2 + \sigma^2 b^2]} \left\{[(\omega - K_m)^2 + L_m]^{-1/2}\right.$$

$$\left. - [(\omega + K_m)^2 + L_m]^{-1/2}\right\} d\omega .$$

2-3 MAIN WAVE INTEGRAL (Case $c_1 > c_2$)

When the velocity of the bottom is less than the velocity of the liquid above the bottom, there can be no precursor. The integration paths are slightly different from the case $c_2 > c_1$ and are shown in Figure 2-3.1 below.

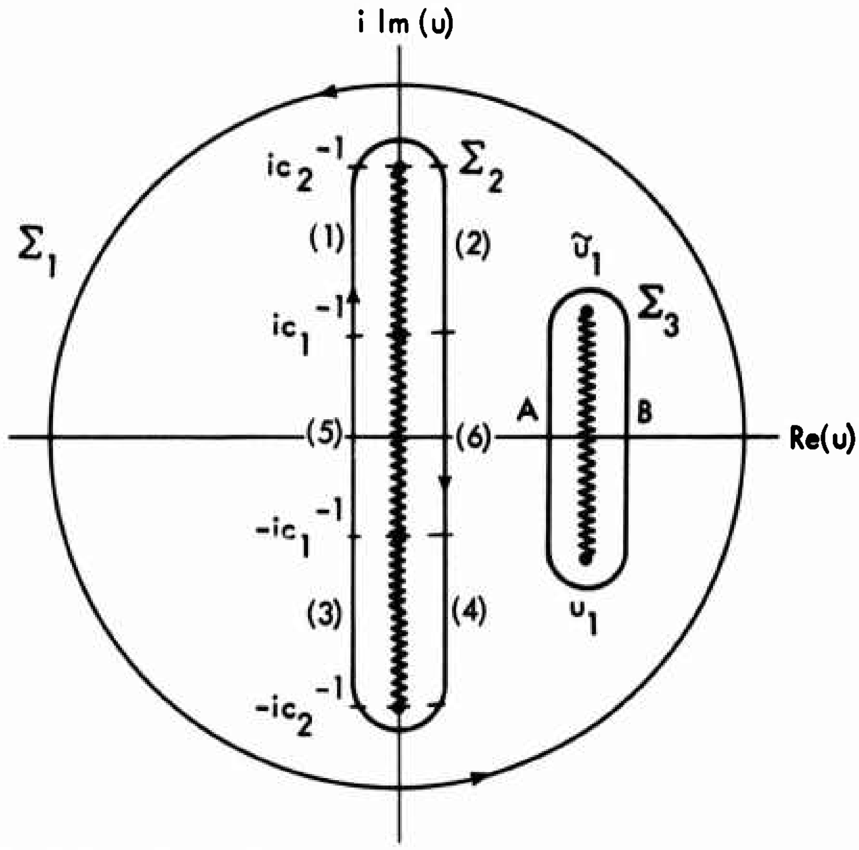

FIG. 2-3.1 PATH OF INTEGRATION IN THE υ-PLANE FOR $\tau > c_1^{-1} R_m$ (LIQUID BOTTOM $c_1 > c_2$)

In comparison with the previous case, α_1 and α_2 are real on paths (5) and (6), and, as before, the integrals on these paths cancel. Thus we can write

$$_nP_m = \frac{(-1)^{m+n}}{R_m}\left(\frac{1-b}{1+b}\right)^n - \frac{1}{2}\oint_{\Sigma} P'du \, . \qquad (2\text{-}3.1)$$

Let the integral on path Σ_2 be denoted I_2. We then replace $-u$ by u on paths (3) and (4) and interchange the limits of integration on paths (2) and (3) to get

$$I_2 = (1)\int_{ic_1^{-1}}^{ic_2^{-1}} P'du - (2)\int_{ic_1^{-1}}^{ic_2^{-1}} P'du - (3)\int_{ic_1^{-1}}^{ic_2^{-1}} P'du \qquad (2\text{-}3.2)$$

$$+ (4)\int_{ic_1^{-1}}^{ic_2^{-1}} P'du .$$

On paths (1), (2), (3), and (4) α_2 is real, but α_1 is imaginary. The liquid bottom reflection coefficient for α_1 imaginary and positive but α_2 real and positive is

$$\bar{K} = \frac{i[-(u^2 + c_1^{-2})]^{1/2} - b\alpha_2}{i[-(u^2 + c_1^{-2})]^{1/2} + b\alpha_2} . \qquad (2\text{-}3.3)$$

Separating real and imaginary parts of $\bar{K}$, we obtain

$$\mathrm{Im}(\bar{K}) = \frac{2b\alpha_2[-(u^2 + c_1^{-2})]^{1/2}}{b^2\alpha_2^2 - (u^2 + c_1^{-2})} \quad \text{and}$$

$$(2\text{-}3.4)$$

$$\mathrm{Re}(\bar{K}) = \frac{-(u^2 + c_1^{-2}) - b^2\alpha_2^2}{b^2\alpha_2^2 - (u^2 + c_1^{-2})} .$$

Let $\bar{\gamma}_1$ and $\bar{\gamma}_2$ denote respectively the values of γ for which (1) α_1 is imaginary and positive and (2) α_1 is imaginary and negative. The square root α_1 , being imaginary, is positive on paths (1) and (2), but negative on paths (3) and (4). Hence $\mathrm{Re}(\bar{K})$ is the same on all the paths, and the contributions from $\mathrm{Re}(\bar{K})$ cancel.

In contrast to the case $c_2 > c_1$, the square root γ is now complex so there may be contributions from both the real and the

imaginary parts. In the contributions from $\mathrm{Im}(\bar{K})$ the imaginary α_1's cancel out the numerator and the denominator of ${}_1P_m$ so that the real contribution to ${}_1P_m$ comes from $\mathrm{Im}(\gamma)$.

The integrals along paths (1) and (2) and along (3) and (4) may be combined to get an equation analogous to equation (2-2.7) when $c_2 > c_1$. Since γ_1 and $\bar{\gamma}_2$ are complex conjugates, $1/\bar{\gamma}_1$ and $1/\bar{\gamma}_2$ are complex conjugates, and there is still symmetry about the real axis. The α_1's may be canceled, and we obtain

$$I_2 = (-1)^m \frac{4b}{\pi} \int_{ic_1^{-1}}^{ic_2^{-1}} \frac{\alpha_2 u}{[b^2\alpha_2^{\,2} - (u^2 + c_1^{\,-2})]} \left\{ \mathrm{Im}\left(\frac{1}{\bar{\gamma}_1}\right) - \mathrm{Im}\left(\frac{1}{\bar{\gamma}_2}\right) \right\} du \;. \qquad (2\text{-}3.5)$$

The substitution $u = ix$ yields

$$I_2 = (-1)^{m+1} \frac{4b}{\pi} \int_{c_1^{-1}}^{c_2^{-1}} \frac{\alpha_2 x}{[b^2\alpha_2^{\,2} + (x^2 - c_1^{\,-2})]} \left\{ \mathrm{Im}\left(\frac{1}{\bar{\gamma}_1}\right) - \mathrm{Im}\left(\frac{1}{\bar{\gamma}_2}\right) \right\} dx \;. \qquad (2\text{-}3.6)$$

The following Rosenbaum substitutions are now used: $\tau_m = \tau/R_m$, $\cos \ell_m = d_m/R_m$, $\sin \ell_m = r/R_m$, $D_m = \tau_m^{\,2} \cos 2\ell_m + c_1^{\,-2} \sin^2 \ell_m$, $E_m = 4(\sin^2 \ell_m \cos^2 \ell_m)\,\tau_m^{\,2}(\tau_m^{\,2} - c_1^{\,-2})$, and $F_m = \tau_m^{\,2} - c_1^{\,-2} \sin^2 \ell_m$. Also let $\bar{\omega} = (x^2 - c_1^{\,-2})^{1/2}$ and $\bar{\sigma} = (c_2^{\,-2} - c_1^{\,-2})^{1/2}$. The limits of integration then become $\bar{\omega}_1 = (c_1^{\,-2} - c_1^{\,-2})^{1/2} = 0$ and $\bar{\omega}_2 = (c_2^{\,-2} - c_1^{\,-2})^{1/2} = \bar{\sigma}$. After applying these substitutions to equation (2-3.6), we obtain

$$I_2 = (-1)^{m+1} \frac{4b}{\pi} \int_0^{\bar{\sigma}} \frac{\bar{\omega}(\bar{\sigma}^2 - \bar{\omega}^2)^{1/2}}{[(1-b^2)\bar{\omega}^2 + \bar{\sigma}^2 b^2]} \left\{ \mathrm{Im}\left(\frac{1}{\bar{\gamma}_1}\right) - \mathrm{Im}\left(\frac{1}{\bar{\gamma}_2}\right) \right\} d\omega \;. \qquad (2\text{-}3.7)$$

The term $\mathrm{Im}(1/\bar{\gamma}_1)$ can be written

$$\frac{1}{\bar{\gamma}_1} =$$

$$\frac{[-(\bar{\omega}^2+D_m)+2\tau_m^2 \cos^2 \ell_m+2\tau_m i\bar{\omega} \cos \ell_m]^{1/2}}{R_m [(\bar{\omega}^2+D_m)^2+E_m]^{1/2}} = \frac{V^{1/2}}{R_m[(\bar{\omega}^2+D_m)^2+E_m]^{1/2}}. \quad (2\text{-}3.8)$$

The complex square root $V^{1/2}$ can be written $V^{1/2} = |V|^{1/2} \{\cos[\theta(\bar{\gamma}_1)/2] + i \sin [\theta(\bar{\gamma}_1)/2]\}$, where

$$\theta(\bar{\gamma}_1) = \arccos \left\{\frac{-(\bar{\omega}^2 + D_m) + 2\tau_m^2 \cos^2 \ell_m}{[(\bar{\omega}^2 + D_m)^2 + E_m]^{1/2}}\right\}. \quad (2\text{-}3.9)$$

Since $\sin (\theta/2) = [(1 - \cos \theta)/2]^{1/2}$, we obtain

$$\mathrm{Im}(V^{1/2}) = \frac{1}{R_m}\left[(\bar{\omega}^2+D_m)^2+E_m\right]^{1/4}\left[\frac{1-\cos\left[\arccos\left\{\frac{-(\bar{\omega}^2+D_m)+2\tau_m^2 \cos^2 \ell_m}{[(\bar{\omega}^2+D_m)^2+E_m]^{1/2}}\right\}\right]}{2}\right]^{1/2}. \quad (2\text{-}3.10)$$

Simplifying, we obtain

$$\mathrm{Im}(V^{1/2}) = \frac{\sqrt{2}}{2R_m}\left\{\left[(\bar{\omega}^2 + D_m)^2 + E_m\right]^{1/2} + (\bar{\omega}^2 - F_m)\right\}^{1/2}. \quad (2\text{-}3.11)$$

The terms $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are complex conjugates so $\theta(\bar{\gamma}_2) = \theta(\bar{\gamma}_1) + \pi$ and $\mathrm{Im}(\frac{1}{\bar{\gamma}_2}) = -\mathrm{Im}(\frac{1}{\bar{\gamma}_1})$.

Hence we can write

$$\left[\mathrm{Im}(\frac{1}{\bar{\gamma}_1}) - \mathrm{Im}(\frac{1}{\bar{\gamma}_2})\right] = \frac{\sqrt{2}}{R_m}\left\{\frac{[(\bar{\omega}^2 + D_m)^2 + E_m]^{1/2} + (\bar{\omega}^2 - F_m)}{(\bar{\omega}^2 + D_m)^2 + E_m}\right\}^{1/2}. \quad (2\text{-}3.12)$$

Substituting equation (2-3.12) into equation (2-3.7), we get

$$I_2 = (-1)^{m+1} \frac{4\sqrt{2}b}{\pi R_m} \int_0^{\bar{\sigma}} \frac{\bar{\omega}(\bar{\sigma}^2-\bar{\omega}^2)^{1/2}}{[(1-b^2)\bar{\omega}^2+\bar{\sigma}^2 b^2]}\left\{\frac{[(\bar{\omega}^2+D_m)^2+E_m]^{1/2}+(\bar{\omega}^2-F_m)}{(\bar{\omega}^2+D_m)^2+E_m}\right\}^{1/2} d\bar{\omega}. \quad (2\text{-}3.13)$$

Equations (2-3.13) and (2-3.1) then give the pressure equation for $c_1 > c_2$:

$$ {}_1P_m(\tau_m) = 0 \qquad (\tau_m < c_1^{-1}) $$

$$ {}_1P_m(\tau_m) = \frac{(-1)^{m+1}}{R_m}\left(\frac{1-b}{1+b}\right) \qquad (\tau_m > c_1^{-1}) $$

$$ + (-1)^m \frac{2\sqrt{2}b}{\pi R_m} \int_0^{\bar{\sigma}} \frac{\bar{\omega}(\bar{\sigma}^2-\bar{\omega}^2)^{1/2}}{[(1-b^2)\bar{\omega}^2+\bar{\sigma}^2 b^2]} \left\{\frac{[(\bar{\omega}^2+D_m)^2+E_m]^{\frac{1}{2}}+(\bar{\omega}^2-F_m)}{(\bar{\omega}^2+D_m)^2+E_m}\right\}^{\frac{1}{2}} d\bar{\omega}. $$

(2-3.14)

The above equation concludes the derivation of Rosenbaum's expressions for ${}_1P_m$ over a liquid bottom. Similar equations for a rigid bottom are derived in Chapter IV.

CHAPTER III

SOLUTION OF THE RIGID BOTTOM WAVE EQUATION

The derivation of the step wave responses $A_1(r,z,\tau)$ and $A_2(r,z,\tau)$ for a rigid bottom is quite similar to that for a liquid bottom and will be treated briefly.

3-1 WAVE EQUATION IN THE WATER

In the water we choose a displacement potential ϕ such that the displacement of an element of fluid is $\vec{s} = \vec{\nabla}\phi$. This potential is related to the velocity potential φ of Chapter I by the equation $\varphi = -\partial\phi/\partial\tau$. Substituting this expression for φ in the velocity potential wave equation (equation (1-1.9) and integrating the resulting equation with respect to τ, the displacement potential satisfies

$$\nabla^2\phi = \frac{1}{c_1{}^2}\frac{\partial^2\phi}{\partial\tau^2}\ , \tag{3-1.1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{3-1.2}$$

and where c_1 is the sound velocity in the water. The potential ϕ and the other potentials used in this chapter are taken to be zero as τ and R approach infinity. The radial and vertical displacements and the pressure are then

$$u_r = \frac{\partial \phi}{\partial r}\ ,\ u_z = \frac{\partial \phi}{\partial z}\ ,\ \text{and}\ \ P_1 = -\rho_1 \frac{\partial^2 \phi}{\partial \tau^2}\ , \tag{3-1.3}$$

where ρ_1 is the density of the water.

3-2 WAVE EQUATION IN THE RIGID BOTTOM

In the rigid bottom* the equations of motion can be expressed in terms of the displacements u_r and u_z in the r and z directions as follows (Ewing, Jardetzky, Press [10]):

$$(\lambda'+2\mu)\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_z}{\partial z \partial r}\right) + \mu\left(\frac{\partial^2 u_r}{\partial z^2} - \frac{\partial^2 u_z}{\partial z \partial r}\right) = \rho_2 \frac{\partial^2 u_r}{\partial \tau^2} \tag{3-2.1}$$

$$(\lambda'+2\mu)\left(\frac{\partial^2 u_r}{\partial z \partial r} + \frac{1}{r}\frac{\partial u_r}{\partial z} + \frac{\partial^2 u_z}{\partial z^2}\right) - \frac{\mu}{r}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)$$

$$-\mu\left(\frac{\partial^2 u_r}{\partial z \partial r} - \frac{\partial^2 u_z}{\partial r^2}\right) = \rho_2 \frac{\partial^2 u_z}{\partial \tau^2}\ , \tag{3-2.2}$$

where λ' and μ are Lamé's elastic constants and ρ_2 is the density of the bottom.

We can define potentials ψ and U such that the radial and vertical displacements (u_r and u_z) and the normal and tangential stresses (T_{zz} and T_{rz}) may be expressed by the following equations:

$$u_r = \frac{\partial \psi}{\partial r} - \frac{\partial U}{\partial z}\ ,\ u_z = \frac{\partial \psi}{\partial z} + \frac{\partial U}{\partial r} + \frac{U}{r} \tag{3-2.3}$$

---

*The rigid bottom solution given here is similar to those given by T. W. Spencer [11], Eichler and Rattayya [9], and Roever, Vining, and Strick [12].

$$T_{zz} = \lambda'\nabla^2\psi + 2\mu\left\{\frac{\partial^2\psi}{\partial z^2} + \frac{1}{r}\frac{\partial^2(rU)}{\partial z\partial r}\right\}, \tag{3-2.4}$$

$$T_{rz} = \mu\left\{2\frac{\partial^2\psi}{\partial r\partial z} - 2\frac{\partial^2 U}{\partial z^2} + \frac{1}{c_4{}^2}\frac{\partial^2 U}{\partial\tau^2}\right\}, \tag{3-2.5}$$

where c_3 and c_4 are respectively the velocities of propagation of sound and shear waves in the bottom. Lamé's constants λ' and μ are related to c_3 and c_4 as follows:

$$c_3 = \left(\frac{\lambda' + 2\mu}{\rho_2}\right)^{1/2} \quad \text{and} \quad c_4 = (\mu/\rho_2)^{1/2}. \tag{3-2.6}$$

Then the potentials ψ and U satisfy the equations

$$\nabla^2\psi = \frac{1}{c_3{}^2}\frac{\partial^2\psi}{\partial\tau^2} \tag{3-2.7}$$

and

$$\nabla^2 U - \frac{U}{r^2} = \frac{1}{c_4{}^2}\frac{\partial^2 U}{\partial\tau^2}. \tag{3-2.8}$$

Equation (3-2.8) may be replaced by a wave equation by using a potential ξ defined by $U = -\partial\xi/\partial r$. We then obtain

$$\nabla^2\xi = \frac{1}{c_4{}^2}\frac{\partial^2\xi}{\partial\tau^2}. \tag{3-2.9}$$

3-3 BOUNDARY CONDITIONS FOR A STEP WAVE SOURCE

As for the liquid bottom we now seek a solution $A_1(r,z,\tau)$ for a pressure source $H(\tau - R/c_1)/R$. The boundary conditions which must be satisfied are as follows:

1. The pressure $A_1(r,z,\tau)$ in the water must vanish at the surface $z = -(D - h)$.

2. The tangential component of stress T_{rz} must vanish at the interface $z = h$.

3. The normal component of displacement u_z must be continuous at the interface $z = h$.

4. The normal component of stress T_{zz} must be continuous at the interface $z = h$.

5. Near the source the solution for $A_1(r,z,\tau)$ must approach $P_A = H(\tau - R/c_1)/R$.

6. As z tends to infinity ψ and U must tend to zero.

3-4 TRANSFORMED POTENTIALS $\bar{\phi}$, $\bar{\psi}$, $\bar{U}$

Using the same Laplace transform method we used in the liquid bottom solution (see equations (1-4.14) and (1-4.15)), we find that the transformed solutions are

$$\bar{\phi} = \int_0^\infty \left\{ f_1(\lambda) \exp(\beta_1 z) + g_1(\lambda) \exp(-\beta_1 z) \right\} J_0(\lambda r) \lambda d\lambda \tag{3-4.1}$$

$$- \frac{1}{\rho_1 s^3} \int_0^\infty \left\{ J_0(\lambda r) \exp(-\beta_1 |z|) \frac{\lambda}{\beta_1} d\lambda \right. ,$$

$$\bar{\psi} = \int_0^\infty g_3(\lambda) \exp(-\beta_3 z) J_0(\lambda r) \lambda d\lambda , \tag{3-4.2}$$

$$\bar{U} = - \frac{\partial \bar{\xi}}{\partial r} = \int_0^\infty g_4(\lambda) \exp(-\beta_4 z) \frac{d}{dr} [J_0(\lambda r)] \lambda d\lambda , \tag{3-4.3}$$

where f_1, g_1, g_3, and g_4 are functions of λ to be determined from a transformed version of the boundary data and where $\beta_i = (s^2 c_i^{-2} + \lambda^2)^{1/2}$ for $i = 1, 3, 4$. The last term of $\bar{\phi}$ is obtained from the transformed

pressure source $\overline{P}_A$ written as a Bessel function integral (equation (1-4.13)) by using the pressure relation $P = -\rho\partial^2\phi/\partial\tau^2$ and its transform $\overline{P} = -s^2\rho\overline{\phi}$.

3-5 TRANSFORMED STEP WAVE RESPONSE IN A RIGID BOTTOM

Since $\overline{A}_1(r,z,s)$ is a transformed pressure, we can write from equation (3-4.1):

$$\overline{A}_1 = -s^2\rho_1\overline{\phi} =$$

$$\int_0^\infty \left\{a_1(\lambda)\exp(\beta_1 z) + b_1(\lambda)\exp(-\beta_1 z)\right\}J_0(\lambda r)\lambda d\lambda + \frac{1}{s}\int_0^\infty J_0(\lambda r)\exp(-\beta_1|z|)\frac{\lambda d\lambda}{\beta_1}, \tag{3-5.1}$$

where $a_1(\lambda)$ and $b_1(\lambda)$ are respectively $-s^2\rho_1 f_1(\lambda)$ and $-s^2\rho_1 g_1(\lambda)$.

The application of the transforms of boundary conditions (1) thru (4) yields the following equations. The requirement of the vanishing of the transformed pressure $\overline{A}_1(r,z,s)$ at the surface, namely

$$\overline{A}_1(r,z,s) = -s^2\rho_1\overline{\phi} = 0 \quad \text{at} \quad z = -(D-h),$$

yields the equation

$$\text{(1)} \quad f_1(\lambda)\exp[-\beta_1(D-h)] + g_1(\lambda)\exp[\beta_1(D-h)] - \frac{1}{\rho_1 s^3\beta_1}\exp[-\beta_1(D-h)] = 0. \tag{3-5.2}$$

The vanishing of the tangential component of stress T_{rz} and its transform $\overline{T}_{rz}$ at the bottom-water interface, namely

$$\overline{T}_{rz} = \mu\left\{2\frac{\partial\overline{\Psi}}{\partial r\partial z} - 2\frac{\partial^2\overline{U}}{\partial z^2} + \frac{s^2}{C_4{}^2}\overline{U}\right\} = 0 \text{ at } z = h,$$

yields the equation

$$2\left\{-\beta_3 \lambda g_3(\lambda)\exp(-\beta_3 h)J_0{}'(\lambda r)\right\} - 2\left\{\beta_4{}^2 \lambda g_4(\lambda)\exp(-\beta_4 h)J_0{}'(\lambda r)\right\}$$

$$+ \frac{s^2}{c_4{}^2}\left\{\lambda g_4(\lambda)\exp(-\beta_4 h)J_0{}'(\lambda r)\right\} = 0, \qquad (3\text{-}5.3)$$

where the prime in $J_0{}'(\lambda r)$ above and in $J_1{}'(\lambda r)$, $J_2{}'(\lambda r)$, and $J_0{}''(\lambda r)$ below denotes the derivatives d/dr and d^2/dr^2. Simplifying the above equation, we obtain

$$(2) \quad 2\beta_3\ g_3(\lambda)\ \exp(-\beta_3 h) + (2\lambda^2 + s^2 c_4{}^{-2})g_4(\lambda)\ \exp(-\beta_4 h) = 0. \qquad (3\text{-}5.4)$$

The continuity of the normal component of displacement u_z and its transform $\bar{u}_z$ at the bottom-water interface, namely

$$\bar{u}_z = \frac{\partial \bar{\Psi}}{\partial z} + \frac{\partial \bar{U}}{\partial r} + \frac{\bar{U}}{r} = \frac{\partial \bar{\phi}}{\partial z} \quad \text{at} \quad z = h ,$$

yields the equation

$$\left\{-\beta_3 \lambda g_3(\lambda)\exp(-\beta_3 h)J_0(\lambda r)\right\} + \left\{\lambda g_4(\lambda)\exp(-\beta_4 h)J_0{}''(\lambda r)\right\} +$$

$$\left\{\frac{\lambda}{r} g_4(\lambda)\exp(-\beta_4 h)J_0{}'(\lambda r)\right\} = \left\{\beta_1 \lambda f_1(\lambda)\exp(\beta_1 h) - \beta_1 \lambda g_1(\lambda)\exp(-\beta_1 h)\right.$$

$$\left. + \frac{\lambda}{\rho s^3}\exp(-\beta_1 h)\right\} J_0(\lambda r) . \qquad (3\text{-}5.5)$$

Simplifying the above equation using the Bessel function recurrence

formulae for J_0' and J_1' listed below, gives

$$(3)\quad -\beta_3\, g_3(\lambda)\, \exp(-\beta_3 h) - \lambda^2 g_4(\lambda)\, \exp(-\beta_4 h)$$

$$= \beta_1 f_1(\lambda)\, \exp(\beta_1 h) - \beta_1 g_1(\lambda)\, \exp(-\beta_1 h) + \frac{1}{\rho_1 s^3} \exp(-\beta_1 h). \tag{3-5.6}$$

The continuity of the normal component of stress T_{zz} and its transform $\overline{T}_{zz}$ at the bottom-water interface, namely

$$\overline{T}_{zz} = -\overline{A}_1(r,z,s) \quad \text{at} \quad z = h,$$

leads to

$$\lambda'\nabla^2\overline{\psi} + 2\mu\left\{\frac{\partial^2\overline{\psi}}{\partial z^2} + \frac{1}{r}\frac{\partial^2(r\overline{U})}{\partial z\partial r}\right\} = s^2\rho_1\overline{\phi} \quad \text{at} \quad z = h\ .$$

Using the transformed wave equation obtained from equation (3-2.7),

$$\nabla^2\overline{\psi} = \frac{s^2}{C_3{}^2}\,\overline{\psi}\ ,$$

we obtain

$$\lambda'\frac{s^2}{C_3{}^2}\left\{\lambda g_3(\lambda)\exp(-\beta_3 h)J_0(\lambda r)\right\} + 2\mu\left\{[\beta_3{}^2\lambda g_3(\lambda)\exp(-\beta_3 h)J_0(\lambda r)]\right.$$

$$\left. + [-\beta_4\frac{\lambda}{r}g_4(\lambda)\exp(-\beta_4 h)J_0'(\lambda r) - \beta_4\lambda g_4(\lambda)\exp(-\beta_4 h)J_0''(\lambda r)]\right\} \tag{3-5.7}$$

$$= s^2\rho_1\left\{\lambda[f_1(\lambda)\exp(\beta_1 h) + g_1(\lambda)\exp(-\beta_1 h)]J_0(\lambda r) - \frac{\lambda}{\rho_1 s^3\beta_1}\exp(-\beta_1 h)J_0(\lambda r)\right\}.$$

Applying the recurrence formulae for $J_0{}'$ and $J_1{}'$ below and using $c_3 = [(\lambda' + 2\mu)/\rho_2]^{1/2}$ and $c_4 = (\mu/\rho_2)^{1/2}$, the above equation can be simplified to the following expression:

$$(4)\quad 2\rho_2 c_4{}^2 \left\{\frac{1}{2}(2\lambda^2 + s^2 c_4{}^{-2}) g_3(\lambda) \exp(-\beta_3 h) + \lambda^2 \beta_4 g_4(\lambda) \exp(-\beta_4 h)\right\}$$

$$= s^2 \rho_1 \left\{f_1(\lambda) \exp(\beta_1 h) + g_1(\lambda) \exp(-\beta_1 h) - \frac{1}{\rho_1 s^2 \beta_1} \exp(-\beta_1 h)\right\} . \tag{3-5.8}$$

<u>Recurrence Formulae</u>

Above we have used the following Bessel function recurrence formulae [13]:

$$J_0(u) - J_2(u) = 2\frac{d}{du}[J_1(u)], \tag{3-5.9}$$

$$J_0(u) + J_2(u) = \frac{2}{u} J_1(u), \tag{3-5.10}$$

and

$$\frac{d}{du}[J_0(u)] = -J_1(u) . \tag{3-5.11}$$

Solving for $f_1(\lambda)$ and $g_1(\lambda)$ from (1) thru (4) and using $\bar{A}_1 = -s^2 \rho_1 \bar{\phi}$, we find

$$a_1(\lambda) = -s^2 \rho_1 \; f_1(\lambda) = \frac{1}{\beta_1 s}\left\{\frac{K[\exp(-2\beta_1 h) - \exp(-2\beta_1 D)]}{1 + K \exp(-2\beta_1 D)}\right\}, \tag{3-5.12}$$

and

$$b_1(\lambda) = -s^2 \rho_1 \; g_1(\lambda) = \frac{-\exp[-2\beta_1 (D - h)]}{\beta_1 s}\left\{\frac{1 + K \exp(-2\beta_1 h)}{1 + K \exp(-2\beta_1 D)}\right\} , \tag{3-5.13}$$

where the reflection coefficient K is now defined

$$K = \frac{\beta_1[(2\lambda^2+s^2c_4^{-2})^2-4\lambda^2\beta_3\beta_4] - b\beta_3 s^4 c_4^{-4}}{\beta_1[(2\lambda^2+s^2c_4^{-2})^2-4\lambda^2\beta_3\beta_4] + b\beta_3 s^4 c_4^{-4}} \; . \tag{3-5.14}$$

3-6 THE STEP WAVE PRESSURE RESPONSE ${}_nP_m$ FOR A RIGID BOTTOM

The coefficients $a_1(\lambda)$ and $b_1(\lambda)$ have exactly the same form as equations (1-4.19) and (1-4.20) for the liquid bottom. The function $\overline{A}_1$ is then in the same form as equation (1-4.24). The expression $|K \exp(-2\beta_1 D)|$ can be shown to be less than one (Spencer [11]) so that the ray expansion is still valid. The derivation leading to equation (1-11.14), written below, can then be duplicated for a rigid bottom provided we use K as defined by equation (3-5.14) above and make the proper branch cuts for the multiple-valued functions involved.

As for the liquid bottom, the step wave pressure response ${}_nP_m$ resulting from ray (n,m) is

$${}_nP_m = 0 \; , \qquad (\tau < {}_n\delta_m)$$

$${}_nP_m = \frac{(-1)^{n+m}}{\pi i} \int_{u_1}^{\tilde{u}_1} u\alpha_1^{-1} K^n [u^2r^2 + (\tau - {}_nd_m\alpha_1)^2]^{-1/2} du \; , \tag{3-6.1}$$

$$(\tau \geq {}_n\delta_m) \; .$$

The transformation $u = \lambda/s$ renders K in the form necessary for substitution in the above equation:

$$K = \frac{\alpha_1[(2u^2+c_4^{-2})^2 - 4u^2\alpha_3\alpha_4] - b\alpha_3 c_4^{-4}}{\alpha_1[(2u^2+c_4^{-2})^2 - 4u^2\alpha_3\alpha_4] + b\alpha_3 c_4^{-4}} \; , \tag{3-6.2}$$

where $\alpha_1 = (u^2 + c_1^{-2})^{1/2}$, $\alpha_3 = (u^2 + c_3^{-2})^{1/2}$, and $\alpha_4 = (u^2 + c_4^{-2})^{1/2}$.

3-7 STONLEY POLES

In the following calculations we must now consider the branch points of α_1 , α_3 , α_4 and γ and the poles $\pm ik$ of the reflection coefficient K. The branch cuts are made in the same manner as before. The poles $\pm ik$, called the Stonley poles, are the zeros of the denominator of K. It has been shown by Spencer [14] that the Stonley poles are beyond the branch points $\pm ic_1^{-1}$, $\pm ic_3^{-1}$, and $\pm ic_4^{-1}$ on the imaginary axis of the u-plane and also that $\pm ik$ are the only poles on the top sheet of the Riemann plane. Poles of this type indicate the existence of a wave. This particular wave is usually called a Stonley wave (Cagniard [15] refers to it as a Scholte wave), and its propagation velocity is $c_{ST} = 1/k$. Stonley waves propagate along the liquid-solid interface in the same manner as Rayleigh waves in a vacuum-solid interface. The Stonley wave has been studied by Scholte [16], [17] and more recently by Roever, Vining and Strick [12].

CHAPTER IV

THE RIGID BOTTOM SOLUTION IN THE FORM OF ROSENBAUM

In this chapter we follow the same procedure used in Chapter II for a liquid bottom, and we frequently refer back to equations in that chapter. As before, we derive using Rosenbaum's substitutions real integral solutions from equation (3-6.1), where now K is given by equation (3-6.2). Since the rigid bottom K is more complicated and since it has the Stonley poles on the top sheet of the Riemann plane, the integrals obtained will be correspondingly more complicated than for the liquid bottom.

4-1 PRECURSOR INTEGRAL (Case $c_3 > c_1 > c_4$)

The integration path for $\tau_m < c_1^{-1}$, which is essentially identical to the liquid bottom path, is shown in Figure 4-1.1 below. (It must be remembered that c_3 has replaced c_2 as the sound velocity of the bottom.) The branch points $\pm ic_4^{-1}$ of α_4 and the Stonley poles $\pm ik$ are outside the integration path.

For n = 1 we can now write from equation (2-1.10)

$$
{}_1P_m = (-1)^m \frac{(\sigma_1 - M_m)}{2R_m} \int_{-1}^{1} \frac{\mathrm{Im}(K)\left[1-\sin\left(\frac{\pi}{2}\psi\right)\right] d\psi}{(\sigma_1-\omega)^{1/2}(\omega-N_m)^{1/2}} , \qquad (4\text{-}1.1)
$$

where Im(K) must be determined and $\sigma_1 = (c_1^{-2} - c_3^{-2})^{1/2}$. The substitution of Im(K) is permitted here since the sign of the integrand is dependent only on the imaginary square root α_3 as the liquid bottom solution was dependent only on α_2. The real roots α_1 and α_4 are

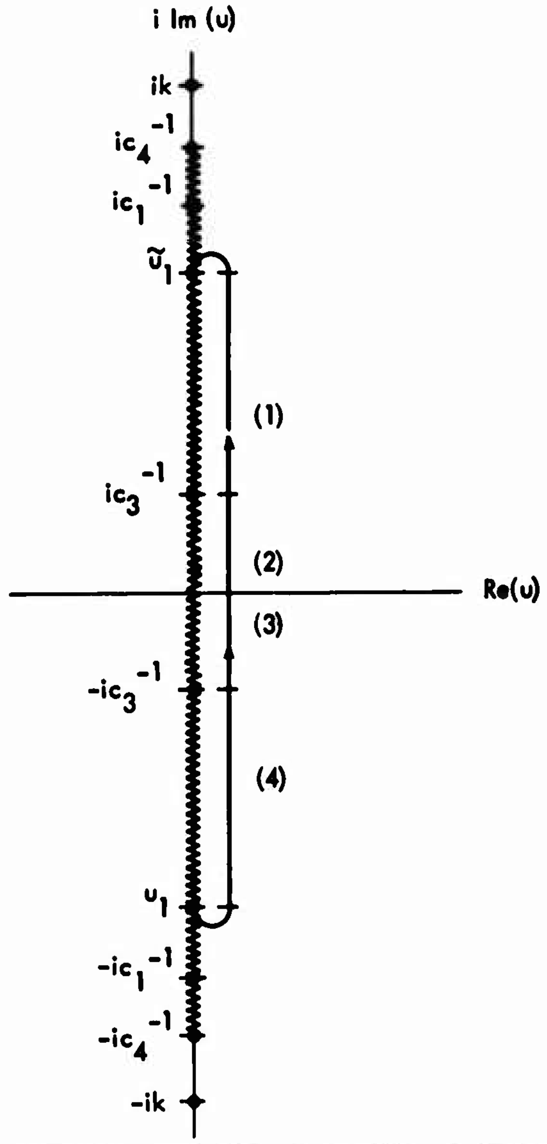

FIG. 4-1.1 PATH OF INTEGRATION IN THE u-PLANE FOR $\tau < c_1^{-1} R_m$ (RIGID BOTTOM $c_3 > c_1 > c_4$)

positive on all the paths. After expanding the imaginary square root α_3, K may be written as follows:

$$K = \frac{\alpha_1\left[(2u^2+c_4^{-2})^2 - 4iu^2(-u^2-c_3^{-2})^{1/2}\alpha_4\right] - bi(-u^2-c_3^{-2})^{1/2}\, c_4^{-4}}{\alpha_1\left[(2u^2+c_4^{-2})^2 - 4iu^2(-u^2-c_3^{-2})^{1/2}\alpha_4\right] + bi(-u^2-c_3^{-2})^{1/2}\, c_4^{-4}}, \tag{4-1.2}$$

where all the square roots in this equation are positive.

The imaginary part of K is then

$$\mathrm{Im}\,(K) = \frac{-2\left[\alpha_1(2u^2+c_4^{-2})^2\right]\left[bc_4^{-4}(-u^2 - c_3^{-2})^{1/2}\right]}{[\alpha_1(2u^2+c_4^{-2})^2]^2+[(-u^2-c_3^{-2})^{1/2}(4u^2\alpha_1\alpha_4-bc_4^{-4})]^2}\ . \tag{4-1.3}$$

The substitutions $\sigma_1 = (c_1^{-2} - c_3^{-2})^{1/2}$, $\sigma_2 = (|c_4^{-2} - c_1^{-2}|)^{1/2}$, and $\omega = (c_1^{-2} - x^2)^{1/2}$ with $u = ix$ yield

$$\mathrm{Im}(K) = \left(-2\left[\omega\left(\frac{c_4^{-2}}{2} - c_1^{-2} + \omega^2\right)^2\right]\left[b\,\frac{c_4^{-4}}{4}(\sigma_1^2 - \omega^2)^{1/2}\right]\right)\Bigg/$$

$$\left(\left[\omega\left(\frac{c_4^{-2}}{2} - c_1^{-2}+\omega^2\right)^2\right]^2+\left\{(\sigma_1^2-\omega^2)^{1/2}\left[\omega(c_1^{-2}-\omega^2)(\sigma_2^2+\omega^2)^{1/2}+\frac{bc_4^{-4}}{4}\right]\right\}^2\right). \tag{4-1.4}$$

The above equations can be further simplified by using

$$A = \omega\left(\frac{c_4^{-2}}{2} - c_1^{-2} + \omega^2\right)^2,$$

$$B = \omega(c_1^{-2} - \omega^2)(\sigma_1^2 - \omega^2)^{1/2}(\sigma_2^2 + \omega^2)^{1/2}$$

$$C = \frac{b}{4}\,c_4^{-4}(\sigma_1^2 - \omega^2)^{1/2}\ .$$

Substituting A, B, and C into Im(K), we get

$$\mathrm{Im}(K) = \frac{-A[b\,c_4^{-4}(\sigma_1^2 - \omega^2)^{1/2}}{2\left\{A^2 + (B + C)^2\right\}} \, . \qquad (4\text{-}1.5)$$

Introducing the above equation into equation (4-1.1), we obtain Rosenbaum's result for $n = 1$ and $c_3 > c_1 > c_4$:

$$_1P_m = 0 \qquad (\tau_m < \delta_m)$$

$$_1P_m(\tau_m) = (-1)^{m+1} \frac{b(\sigma_1 - M_m)}{4R_m\, c_4^4} \int_{-1}^{1} \frac{(\sigma_1 + \omega)^{1/2} A[1 - \sin(\frac{\pi}{2}\,\psi)]\, d\psi}{(\omega - N_m)^{1/2}[A^2 + (B+C)^2]} \, . \qquad (4\text{-}1.6)$$

$$(\delta_m \leq \tau_m < c_1^{-1})$$

4-2 PRECURSOR INTEGRAL (Case $c_3 > c_4 > c_1$)

The case $c_3 > c_4 > c_1$ is not derived by Rosenbaum, but its importance in current work justifies treatment here. The integration path (a) shown in Figure 4-2.1 is chosen when $|u_1| \le c_4^{-1}$; and the path (b), also shown in Figure 4-2.1, is chosen when $|u_1| > c_4^{-1}$. The integral for $|u_1| \le c_4^{-1}$ is almost identical to equation (4-1.6). The only difference is that $c_1^{-1} > c_4^{-1}$ requires replacing B by $B_2 = \omega(c_1^{-2} - \omega^2)(\sigma_1^2 - \omega^2)^{1/2}(|\omega^2 - \sigma_2^2|)^{1/2}$ since now $\sigma_2 = (|c_4^{-2} - c_1^{-2}|)^{1/2} = (c_1^{-2} - c_4^{-2})^{1/2}$.

Thus for $n = 1$, $c_3 > c_4 > c_1$, and $|u_1| \le c_4^{-1}$ we obtain

$$ {}_1P_m(\tau_m) = 0 \qquad (\tau_m < \delta_m)$$

$$ {}_1P_m(\tau_m) = (-1)^{m+1} \frac{b(\sigma_1 - M_m)}{4R_m c_4^4} \int_{-1}^{1} \frac{(\sigma_1 + \omega)^{1/2} A[1-\sin(\frac{\pi}{2}\psi)]\, d\psi}{(\omega - N_m)^{1/2}[A^2 + (B_2 + C)^2]} . \qquad (4\text{-}2.1)$$

$$(\delta_m < \tau_m < c_1^{-1})$$

When $|u_1| > c_4^{-1}$ the integral over paths (3) and (4) is again zero, and ${}_1P_m$ is made up of two parts: 1. The integrals over paths (2) and (5) whose sum is identical to equation (4-2.1) except the lower limit of integration must be replaced by $\psi_1 = \psi(u = ic_4^{-1})$. 2. The integrals over paths (1) and (6) on which α_3 and α_4 are imaginary.

Derivation of the first part of ${}_1P_m$ requires only finding ψ_1. The variable ψ is given by the expression

$$\omega = \frac{(\sigma_1 + M_m)}{2} + \frac{(\sigma_1 - M_m)}{2} \sin \pi\psi/2 . \qquad (4\text{-}2.2)$$

But at $u = ic_4^{-1}$ this expression for ω yields

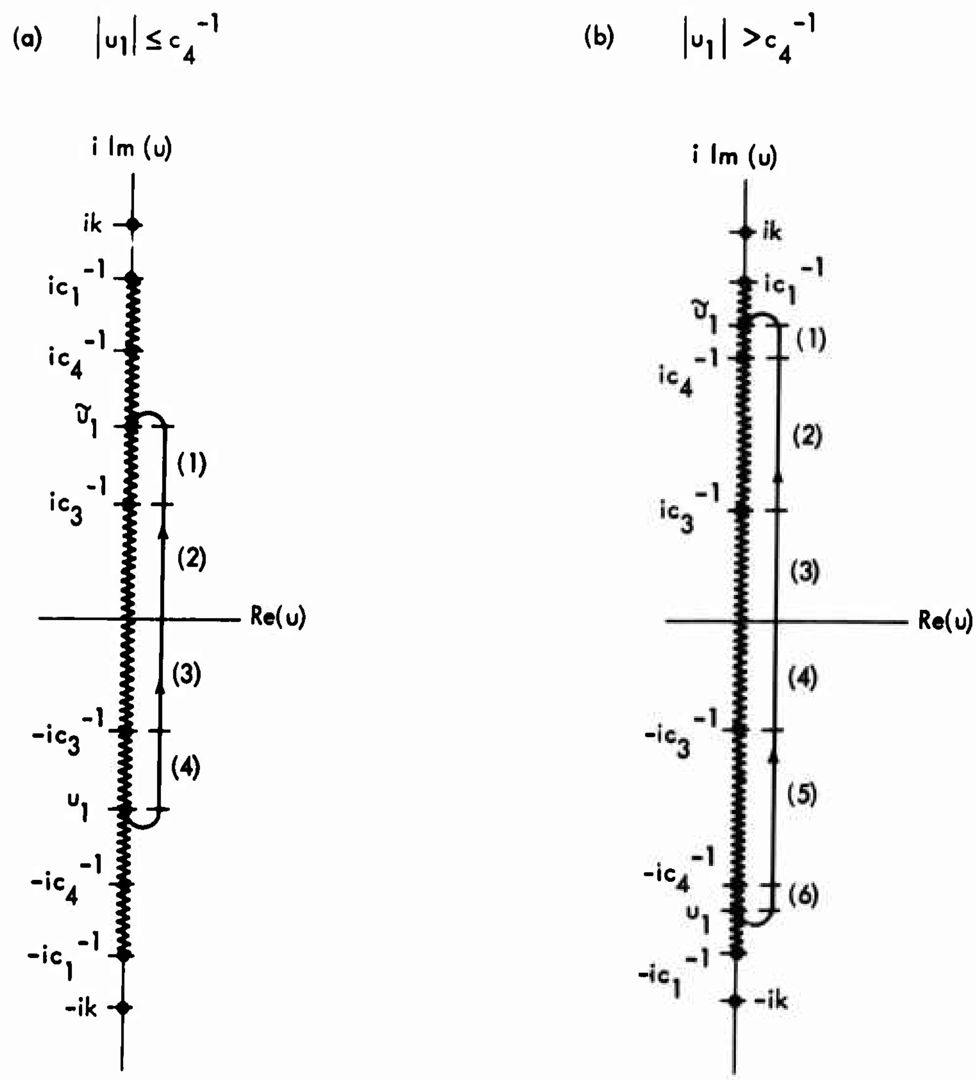

FIG. 4-2.1 PATHS OF INTEGRATION IN THE υ-PLANE FOR $\tau < c_1^{-1} R_m$ (RIGID BOTTOM $c_3 > c_4 > c_1$)

$$\omega_1 = (c_1^{-2} - c_4^{-2})^{1/2} = (|c_4^{-2} - c_1^{-2}|)^{1/2} = \sigma_2 . \qquad (4\text{-}2.3)$$

Then we obtain

$$\sin\frac{\pi\psi_1}{2} = \frac{2\sigma_2 - \sigma_1 - M_m}{\sigma_1 - M_m} \quad \text{and} \quad \psi_1 = \frac{2}{\pi}\arcsin\left[\frac{2\sigma_2 - \sigma_1 - M_m}{\sigma_1 - M_m}\right]. \qquad (4\text{-}2.4)$$

To find the second part of ${}_1P_m$, we must examine K, the only term that contains α_3 and α_4. For α_3 and α_4 imaginary and positive, denote the reflection coefficient by $\bar{\bar{K}}$. Then after expanding the imaginary square roots α_3 and α_4, we obtain

$$\bar{\bar{K}} = \Big(\alpha_1 [(2u^2+c_4^{-2})^2 + 4u^2(-u^2-c_3^{-2})^{1/2}(-u^2-c_4^{-2})^{1/2}] - ibc_4^{-4}(-u^2-c_3^{-2})^{1/2}\Big)\Big/\Big(\alpha_1 [(2u^2+c_4^{-2})^2 + 4u^2(-u^2-c_3^{-2})^{1/2}(-u^2-c_4^{-2})^{1/2}] + ibc_4^{-4}(-u^2-c_3^{-2})^{1/2}\Big) \qquad (4\text{-}2.5)$$

and

$$\mathrm{Im}(\bar{\bar{K}}) = \Big(-2bc_4^{-4}(-u^2c_3^{-2})^{1/2} \{\alpha_1 [(2u^2+c_4^{-2})^2 + 4u^2(-u^2-c_3^{-2})^{1/2}(-u^2-c_4^{-2})^{1/2}]\}\Big)\Big/ \Big(\{\alpha_1 [(2u^2+c_4^{-2})^2 + 4u^2(-u^2-c_3^{-2})^{1/2}(-u^2-c_4^{-2})^{1/2}]\}^2 + [bc_4^{-4}(-u^2-c_3^{-2})^{1/2}]^2\Big) . \qquad (4\text{-}2.6)$$

Make the substitutions

$$A = \omega\left[\frac{c_4^{-2}}{2} - c_1^{-2} + \omega^2\right]^2 ,$$

$$B_2 = \omega(c_1^{-2} - \omega^2)(\sigma_1^2 - \omega^2)^{1/2}(|\omega^2 - \sigma_2^2|)^{1/2} ,$$

and

$$C = \frac{bc_4^{-4}}{4}(\sigma_1^2 - \omega^2)^{1/2} ,$$

where ω, σ_1, and σ_2 are defined as above. Then divide the numerator and denominator by 16 to get

$$\mathrm{Im}(\bar{\bar{K}}) = - \frac{bc_4^{-4}(\sigma_1^2-\omega^2)^{1/2}(A-B_2)}{2(A-B_2)^2+C^2} . \qquad (4\text{-}2.7)$$

From equation (4-1.2) we can see that on path (6) α_3 and α_4 being negative has the effect of only changing the sign of Im(K) since α_1 is positive on both paths (1) and (6). Hence, as before, the integrals on paths (1) and (6) add. The second part of ${}_1P_m$ can thus be obtained from equation (2-1.10) by substituting $\mathrm{Im}(\bar{\bar{K}})$ from equation (4-2.7) above and changing the limits of integration.

For $n = 1$, $c_3 > c_4 > c_1$ and $|u_1| > c_4^{-1}$:

$${}_1P_m(\tau_m) = 0 \qquad (\tau_m < \delta_m)$$

$${}_1P_m(\tau_m) = \frac{(-1)^{m+1}b(\sigma_1 - M_m)}{4\,R_m\,c_4^4}\int_{\psi_1}^{1}\frac{(\sigma_1+\omega)^{1/2}A[1-\sin(\frac{\pi}{2}\,\psi)]\,d\psi}{(\omega-N_m)^{1/2}[A^2+(B_2+C)^2]}$$

$$+\ \frac{(-1)^{m+1}b(\sigma_1 - M_m)}{4\,R_m\,c_4^4}\int_{-1}^{\psi_1}\frac{(\sigma_1+\omega)^{1/2}(A-B_2)[1-\sin(\frac{\pi}{2}\,\psi)]\,d\psi}{(\omega-N_m)^{1/2}[(A-B_2)^2 + C^2]} . \quad (\delta_m \le \tau_m < c_1^{-1})$$

$$(4\text{-}2.8)$$

4-3 MAIN WAVE INTEGRAL (Case $c_3 > c_1 > c_4$)

Since this case is more involved than the previous cases a more detailed account will be given. For $\tau_m > c_1^{-1}$ we have seen that u_1 and $\tilde{u}_1$ are in the first and fourth quadrants respectively, and we can replace the integral on path $u_1 A \tilde{u}_1$ by

$$1/2 \int_{\Sigma_3} {}_nP'_m \, du \; .$$

By Cauchy's theorem we can replace Σ_3 by Σ_1, Σ_2, Σ_4, and Σ_5 which are shown in Figure 4-3.1. The path Σ_1, is again a circle whose radius tends to infinity. Thus, we have

$${}_nP_m = -\, 1/2 \left\{ \int_{\Sigma_1} {}_nP'_m \, du + \int_{\Sigma_2} {}_nP'_m \, du + \int_{\Sigma_4} {}_nP'_m \, du + \int_{\Sigma_5} {}_nP'_m \, du \right\}. \qquad (4\text{-}3.1)$$

Applying the Residue Theorem, we obtain

$${}_nP_m = -\pi i \, [R_\infty - (R_{ik} + R_{-ik})] - 1/2 \int_{\Sigma_2} {}_nP'_m \, du \; , \qquad (4\text{-}3.2)$$

where R_∞, R_{ik}, and R_{-ik} are the residues of ${}_nP'_m$ at ∞, ik, and -ik respectively and are derived in Appendix A. The negative sign immediately before $R_{ik} + R_{-ik}$ appears because the contours Σ_4 and Σ_5 are taken in the negative (clockwise) direction. The details of the above procedure are contained in Appendix A.

The branch cuts will be the usual ones. The square roots α_1, α_3, and α_4 have positive real parts in the right half plane, negative real parts in the left half plane, positive imaginary parts in the top half plane, and negative imaginary parts in the bottom half

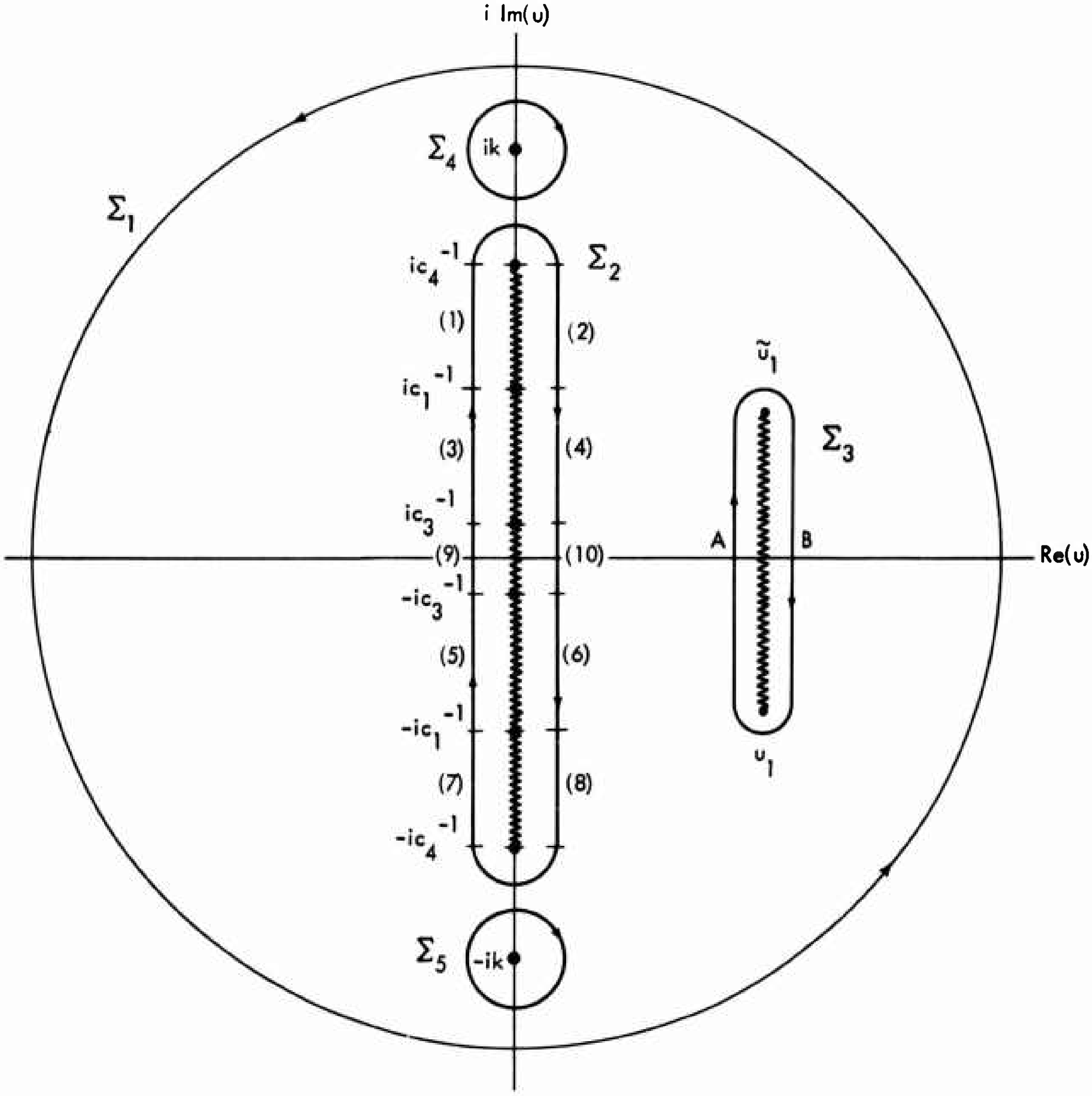

FIG. 4-3.1 PATH OF INTEGRATION IN THE u-PLANE FOR $\tau > c_1^{-1}R_m$ (RIGID BOTTOM $c_3 > c_1 > c_4$)

plane. The function γ has a positive real part left of the cut from u_1 to $\tilde{u}_1$ and a negative real part right of the cut.

The signs on the square roots then require division of the branch line integral (again denoted by I_2) into the ten paths shown in Figure 4.3-1. Since c_3 is the largest velocity, the integrals between $-ic_3^{-1}$ and ic_3^{-1} cancel. We are then left with integrals along paths (1) through (8). Converting negative limits of integration to positive and interchanging the limits of integration when needed, I_2 can be written as follows:

$$I_2 = (1)\int_{ic_1^{-1}}^{ic_4^{-1}} {}_nP'_m\, du - (2)\int_{ic_1^{-1}}^{ic_4^{-1}} {}_nP'_m\, du - (3)\int_{ic_1^{-1}}^{ic_3^{-1}} {}_nP'_m\, du$$

$$+ (4)\int_{ic_1^{-1}}^{ic_3^{-1}} {}_nP'_m\, du + (5)\int_{ic_1^{-1}}^{ic_3^{-1}} {}_nP'_m\, du - (6)\int_{ic_1^{-1}}^{ic_3^{-1}} {}_nP'_m\, du$$

$$- (7)\int_{ic_1^{-1}}^{ic_4^{-1}} {}_nP'_m\, du + (8)\int_{ic_1^{-1}}^{ic_4^{-1}} {}_nP'_m\, du \,, \tag{4-3.3}$$

where the tags are retained so that the proper signs can be placed on the square roots.

The above paths may be grouped into two sets: (a) paths (1), (2), (7), and (8); and (b) paths (3), (4), (5), and (6). In set (a) the limits of integration are ic_1^{-1} to ic_4^{-1}, and α_1 is imaginary. In set (b) the limits are ic_3^{-1} to ic_1^{-1}, and α_1 is real.

Denote the reflection coefficient in set (a) by $\overline{K}$, and let $\overline{\gamma}_1$ and $\overline{\gamma}_2$ denote respectively γ when α_1 is (1) imaginary and positive

and (2) imaginary and negative. Similarly in set (b) let γ_1 and γ_2 denote respectively γ for α_1; (1) real and positive and (2) real and negative. Let $I_2 = I_2' + I_2''$, where I_2' is the contribution from set (a) and I_2'' is the contribution from set (b).

The integrals in set (b) correspond to the integrals of the liquid bottom main wave (case $c_2 > c_1$) with σ and Im(K) being replaced by their rigid bottom counterparts in equation (2-2.9). Equation (2-2.9) can be written for n = 1 in the form

$$I_2' = (-1)^{m+1} \frac{2}{\pi R_m} \int_0^{\sigma_1} \mathrm{Im}(K)\left\{[(\omega-K_m)^2+L_m]^{-1/2}-[(\omega+K_m)^2+L_m]^{-1/2}\right\} d\omega \, . \tag{4-3.4}$$

In set (b) K is given by equation (4-1.2) and Im(K) is given by equation (4-1.3) in terms of u or by equation (4-1.5) in terms of ω, A, B, and C. The sign on the integral is determined by Im(α_3); hence, the integrals on paths (3) and (5) and on paths (4) and (6) add. Using the substitutions defined for the precursor, the contribution to I_2 from set (b) is

$$I_2' = (-1)^m \frac{b}{\pi R_m c_4^{\,4}} \int_0^{\sigma_1} \frac{A(\sigma_1^{\,2}-\omega^2)^{1/2}}{[A^2+(B+C)^2]} \left\{\frac{1}{[(\omega-K_m)^2+L_m]^{1/2}} - \frac{1}{[(\omega+K_m)^2+L_m]^{1/2}}\right\} d\omega \, . \tag{4-3.5}$$

Since α_1 is imaginary, the integrals in set (a) correspond to those of the liquid bottom main wave (case $c_1 > c_2$) with σ and Im(K) being replaced by their solid bottom counterparts. Taking the square roots α_1, α_3, and α_4 to be positive, we can write using only real square roots

$$\bar{K} = \Big(i(-u^2-c_1{}^{-2})^{1/2}[(2u^2+c_4{}^{-2})^2 - 4iu^2(-u^2-c_3{}^{-2})^{1/2}\alpha_4]$$

$$- bi(-u^2-c_3{}^{-2})^{1/2}c_4{}^{-4}\Big)/\Big(i(-u^2-c_1{}^{-2})^{1/2}[(2u^2+c_4{}^{-2})^2 \qquad (4\text{-}3.6)$$

$$- 4iu^2(-u^2-c_3{}^{-2})^{1/2}\alpha_4] + bi(-u^2-c_3{}^{-2})^{1/2}c_4{}^{-4}\Big) .$$

Then the imaginary part is

$$\mathrm{Im}(\bar{K}) = \Big(-2[4u^2(-u^2-c_1{}^{-2})^{1/2}(-u^2-c_3{}^{-2})^{1/2}\alpha_4][b(-u^2-c_3{}^{-2})^{1/2}c_4{}^{-4}]\Big)/$$

$$\Big([4u^2(-u^2-c_1{}^{-2})^{1/2}(-u^2-c_3{}^{-2})^{1/2}\alpha_4]^2$$

$$+ [(-u^2-c_1{}^{-2})^{1/2}(2u^2+c_4{}^{-2})^2+b(-u^2-c_3{}^{-2})^{1/2}c_4{}^{-4}]^2\Big) . \qquad (4\text{-}3.7)$$

Making the substitutions $u = ix$, $\bar{\omega} = (x^2-c_1{}^{-2})^{1/2}$, $\sigma_1 = (c_1{}^{-2}-c_3{}^{-2})^{1/2}$, and $\sigma_2 = (|c_4{}^{-2}-c_1{}^{-2}|)^{1/2}$, and dividing numerator and denominator of equation (4-3.7) by 16, we obtain

$$\mathrm{Im}(\bar{K}) = \Big(2[\bar{\omega}(c_1{}^{-2}+\bar{\omega}^2)(\sigma_1{}^2+\bar{\omega}^2)^{1/2}(\sigma_2{}^2-\bar{\omega}^2)^{1/2}][\frac{bc_4{}^{-4}}{4}(\sigma_1{}^2+\bar{\omega}^2)^{1/2}]\Big)/$$

$$\Big([\bar{\omega}(c_1{}^{-2}+\bar{\omega}^2)(\sigma_1{}^2+\bar{\omega}^2)^{1/2}(\sigma_2{}^2-\bar{\omega}^2)^{1/2}]^2 \qquad (4\text{-}3.8)$$

$$+ [\bar{\omega}(\frac{c_4{}^{-2}}{2} - c_1{}^{-2}-\bar{\omega}^2)^2+\frac{bc_4{}^{-4}}{4}(\sigma_1{}^2+\bar{\omega}^2)^{1/2}]^2\Big) .$$

Let $$\bar{A} = \bar{\omega}\Big[\frac{c_4{}^{-2}}{2} - c_1{}^{-2} - \bar{\omega}^2\Big]^2 ,$$

$$\bar{B} = \bar{\omega}(c_1{}^{-2}+\bar{\omega}^2)(\sigma_1{}^2+\bar{\omega}^2)^{1/2}(\sigma_2{}^2-\bar{\omega}^2)^{1/2} ,$$

and

$$\bar{C} = \frac{bc_4{}^{-4}}{4}(\sigma_1{}^2+\bar{\omega}^2)^{1/2} .$$

Then we can write

$$\mathrm{Im}(\bar{K}) = \frac{\bar{B}}{(\bar{A}+\bar{C})^2+\bar{B}^2}\left[\frac{bc_4{}^{-4}}{2}(\bar{\omega}^2+\sigma_1{}^2)^{1/2}\right] . \qquad (4\text{-}3.9)$$

Similarly, the $\mathrm{Re}(\bar{K})$ can be written

$$\mathrm{Re}(\bar{K}) = \frac{\bar{A}^2+\bar{B}^2-\bar{C}^2}{(\bar{A}+\bar{C})^2+\bar{B}^2} . \qquad (4\text{-}3.10)$$

From equation (4-3.6) one can see that the sign on the integrand ${}_1P'_m$ is determined by α_4 and $\mathrm{Im}(\gamma)$. Hence the integrals along paths (1), (2) and along paths (7) and (8) add. The contribution I_2'' to I_2 can now be written

$$I_2'' = (-1)^{m+1}\frac{2}{\pi}\int_{c_1{}^{-1}}^{c_4{}^{-1}}\frac{\mathrm{Im}(\bar{K})\,x\,dx}{\bar{\omega}}\left[\mathrm{Im}\left(\frac{1}{\gamma_1}\right)-\mathrm{Im}\left(\frac{1}{\gamma_2}\right)\right] . \qquad (4\text{-}3.11)$$

Changing the variable of integration to $\bar{\omega}$, substituting $\mathrm{Im}(\bar{K})$ from equation (4-3.9), and substituting

$$\left[\mathrm{Im}\left(\frac{1}{\gamma_1}\right)-\mathrm{Im}\left(\frac{1}{\gamma_2}\right)\right]$$

from equation (2-3.12), the contribution I_2'' from the set (a) is

$$I_2'' = \frac{(-1)^{m+1}\sqrt{2}\, b}{\pi R_m\, c_4^{\,4}} \int_0^{\sigma_2} \frac{(\bar{\omega}^2+\sigma_1^{\,2})^{1/2}\bar{B}}{[(\bar{A}+\bar{C})^2+\bar{B}^2]} \left\{\frac{\left[(\bar{\omega}^2+D_m)^2+F_n\right]^{1/2}+(\bar{\omega}^2-F_m)}{[(\bar{\omega}^2+D_m)^2+E_m]}\right\}^{1/2} d\bar{\omega} \; . \tag{4-3.12}$$

Substituting R_∞ and $(R_{ik}+R_{-ik})$ from Appendix A and I_2' and I_2'' from equations (4-3.5) and (4-3.12) into equation (4-3.2), we get the pressure equation for $n = 1$, $\tau_m > c_1^{\,-1}$, and $c_3 > c_1 > c_4$:

$$
\begin{aligned}
{}_1P_m &= \frac{1}{R_m} + {}_1\Delta_m \\
&+ \frac{b(-1)^{m+1}}{2\pi R_m\, c_4^{\,4}} \int_0^{\sigma_1} \frac{A(\sigma_1^{\,2}-\omega^2)^{1/2}}{[A^2+(B+C)^2]} \left\{\frac{1}{[(\omega-K_m)^2+L_m]^{1/2}} - \frac{1}{[(\omega+K_m)^2+L_m]^{1/2}}\right\} d\omega \\
&+ \frac{\sqrt{2}\, b(-1)^m}{2\pi R_m\, c_4^{\,4}} \int_0^{\sigma_2} \frac{(\bar{\omega}^2+\sigma_1^{\,2})^{1/2}\bar{B}}{[(\bar{A}+\bar{C})^2+\bar{B}^2]} \left\{\frac{[(\bar{\omega}^2+D_m)^2+E_m]^{1/2}+(\bar{\omega}^2-F_m)}{[(\bar{\omega}^2+D_m)^2+E_m]}\right\}^{1/2} d\bar{\omega} \; ,
\end{aligned}
\tag{4-3.13}$$

where $R_\infty = -\,1/\pi i R_m$ and the contribution ${}_1\Delta_m$ from the Stonley poles is

$${}_1\Delta_m = (-1)^m \frac{\sqrt{2}\, k}{R_m g_1} \left\{\frac{(a^2+f)^{1/2}-a}{a^2+f}\right\}^{1/2} \Gamma \tag{4-3.14}$$

with

$$\Gamma = \left(g_1\left[\left(\frac{c_4^{\,-2}}{2}-k^2\right)^2 - k^2 g_3 g_4\right] - \frac{b g_3}{4c_4^{\,4}}\right) \Bigg/ \left(\frac{k}{g_1}\left[\left(\frac{c_4^{\,-2}}{2}-k^2\right)^2 - k^2 g_3 g_4\right] - g_1 k\left[4\left(\frac{c_4^{\,-2}}{2}-k^2\right) + 2g_3 g_4 + k^2\left(\frac{g_4}{g_3}+\frac{g_3}{g_4}\right)\right] + \frac{bk}{4g_3 c_4^{\,4}}\right), \tag{4-3.15}$$

$g_1 = (k^2 - c_1^{\,-2})^{1/2}$, $g_3 = (k^2 - c_3^{\,-2})^{1/2}$, $g_4 = (k^2 - c_4^{\,-2})^{1/2}$,
$a = \tau_m^2 - (k^2 - c_1^{\,-2}\cos^2\ell_m)$, and $f = 4\tau_m^2\, g_1^{\,2}\cos^2\ell_m$.

It should be noted that equation (4-3.13) differs from Rosenbaum's results by a factor of 2 in each of the two integrals of equation (4-3.13).

4-4 MAIN WAVE INTEGRAL (Case $c_3 > c_4 > c_1$)

When $c_3 > c_4 > c_1$ the integration path is that shown in Figure 4-4.1. The branch cuts are the usual ones along the imaginary axis and from u_1 and $\tilde{u}_1$. As in the previous case, we can write using Cauchy's theorem and the Residue theorem

$$ {}_nP_m = -\pi i \left[R_\infty - (R_{ik} + R_{-ik})\right] - \frac{1}{2} I_8 \,, \qquad (4\text{-}4.1)$$

where

$$ I_8 = \oint_{c_8} {}_nP'_m \, du \,. $$

The integrals on paths (9) and (10) again cancel. Along paths (1), (2), (7), and (8) α_3 and α_4 are imaginary and α_1 is real; but along paths (3), (4), (5), and (6) α_1 and α_4 are real and only α_3 is imaginary. Using the procedure and notation used in the previous case ($c_3 > c_1 > c_4$) the integrals along paths (1) through (8) can be reduced to the following for n = 1:

$$ \begin{aligned} I_8 = {} & \frac{2(-1)^{m+1}}{\pi} \int_{ic_1^{-1}}^{ic_4^{-1}} \frac{\mathrm{Im}(\overset{=}{K})\,udu}{\alpha_1} \left[\frac{1}{\gamma_1} - \frac{1}{\gamma_8}\right] + \\ & \frac{2(-1)^{m+1}}{\pi} \int_{ic_4^{-1}}^{ic_3^{-1}} \frac{\mathrm{Im}(K)\,udu}{\alpha_1} \left[\frac{1}{\gamma_1} - \frac{1}{\gamma_8}\right] \end{aligned} \qquad (4\text{-}4.2)$$

where K is given by equation (4-1.2) and $\overset{=}{K}$ is given by equation (4-2.5). Since $c_4 > c_1$ we must replace B in K by B_8.

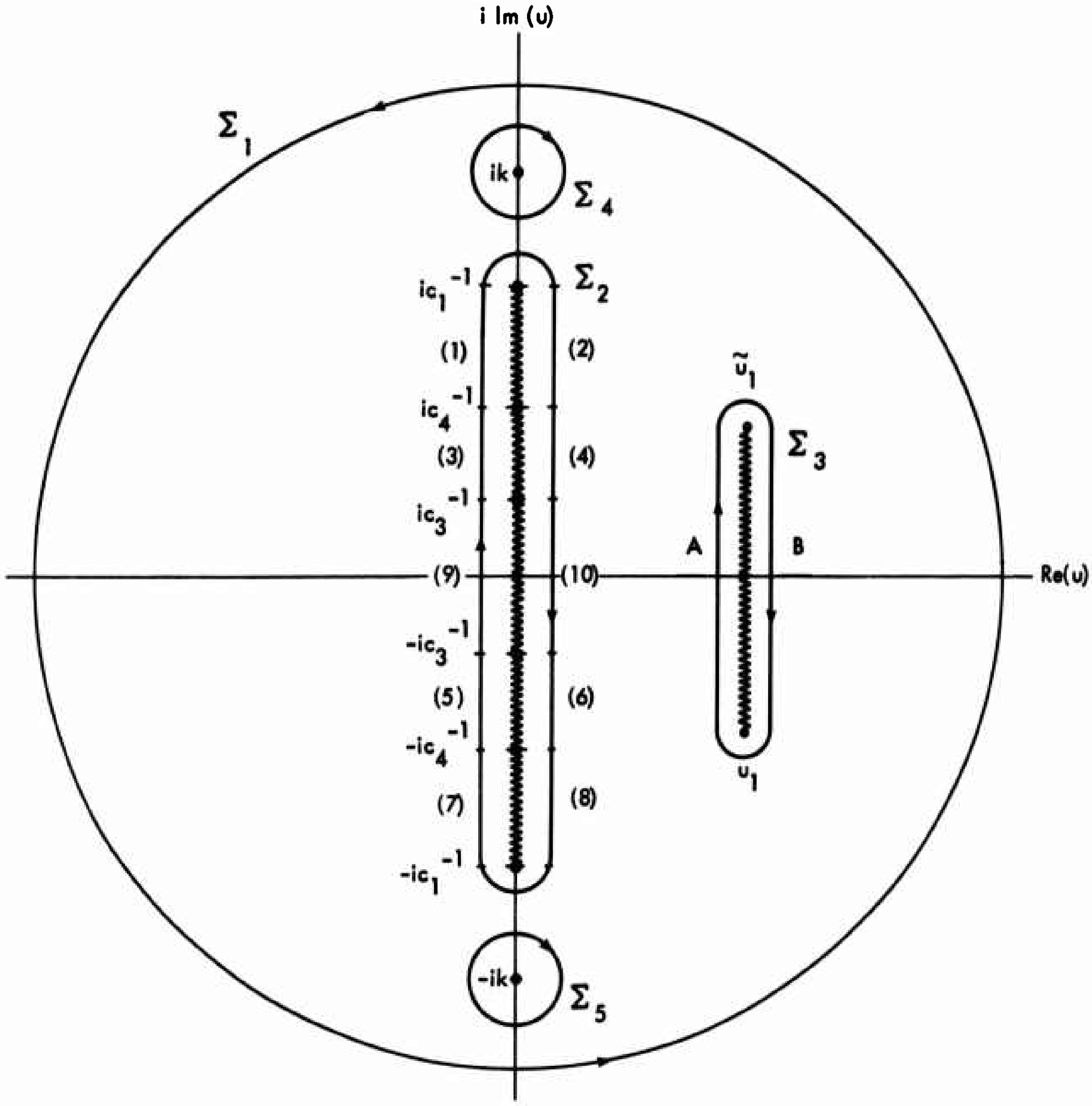

FIG. 4-4.1 PATH OF INTEGRATION IN THE υ-PLANE FOR $\tau > c_1^{-1} R_m$ (RIGID BOTTOM $c_2 > c_4 > c_1$)

The second integral above is given by equation (4-3.5) except the lower limit of integration must be changed to $\sigma_2 = (|c_4^{-2} - c_1^{-2}|)^{1/2}$, the value of ω at $u = ic_4^{-1}$, and B must be replaced by B_2.

The first integral of I_2 is also quite similar to equation (4-3.5). The upper limit of integration must be changed to σ_2, and the imaginary part of the reflection coefficient must be $\mathrm{Im}(\bar{\bar{K}})$ which is given by equation (4-2.7).

Since the residues are unchanged, we may then write for $\tau_m > c_1^{-1}$ and $c_3 > c_4 > c_1$

$$
\begin{aligned}
{}_1P_m &= \frac{1}{R_m} + {}_1\Delta_m \\
&+ \frac{(-1)^{m+1}b}{2\pi R_m\, c_4^{\,4}} \int_0^{\sigma_2} \frac{(\sigma_1^{\,2}-\omega^2)^{1/2}(A-B_2)}{[(A-B_2)^2 + C^2]} \left\{ \frac{1}{[(\omega-K_m)^2+L_m]^{1/2}} - \frac{1}{[(\omega+K_m)^2+L_m]^{1/2}} \right\} d\omega \\
&+ \frac{(-1)^{m+1}b}{2\pi R_m\, c_4^{\,4}} \int_{\sigma_2}^{\sigma_1} \frac{A(\sigma_1^{\,2}-\omega^2)^{1/2}}{[A^2+(B_2+C)^2]} \left\{ \frac{1}{[(\omega-K_m)^2+L_m]^{1/2}} - \frac{1}{[(\omega+K_m)^2+L_m]^{1/2}} \right\} d\omega ,
\end{aligned}
\tag{4-4.3}
$$

where ${}_1\Delta_m$ is given by equations (4-3.14) and (4-3.15).

The above equation completes the derivation of expressions for ${}_1P_m$ in Rosenbaum's form for the most important cases of a rigid bottom. An alternative method of evaluating ${}_nP_m$ which is applicable for all values of n is discussed in the next chapter.

CHAPTER V

THE USE OF COMPLEX ARITHMETIC TO CALCULATE PRESSURE TIME HISTORIES

Although Rosenbaum's equations are in a form quite suitable for numerical calculations, some difficulty is encountered when one wishes to extend the solid bottom solution to n > 1 because of the complexity of the Stonley residues. Also, for all cases n > 1 most of the advantage gained by separating real and imaginary parts is already lost since the equations are so complicated. We then consider evaluating equation (1-11.14) using complex computer arithmetic, as suggested by Eichler and Rattayya [9].

5-1 INTEGRATION PATHS IN THE COMPLEX u-PLANE

Equation (1-11.16),

$$ {}_nP_m = \frac{(-1)^{n+m}}{\pi i} \int_{u_1}^{\tilde{u}_1} u\alpha_1^{-1} K^n \gamma^{-1} \, du \ , \qquad (5\text{-}1.1) $$

where $\gamma = [u^2 r^2 + (\tau - {}_nd_m\,\alpha_1)^2]^{1/2}$ and $u_1 = {}_nR_m^{-2}[-i\tau r + {}_nd_m(\tau^2 - c_1^{-2}\,{}_nR_m^2)^{1/2}]$, can be evaluated using complex computer arithmetic if the integration is divided into the two time periods $\tau < c_1^{-1}\,{}_nR_m$ and $\tau > c_1^{-1}\,{}_nR_m$ which result because of the square root $(\tau^2 - c_1^{-2}\,{}_nR_m^2)^{1/2}$ in u_1. In either case we can evaluate the integral over half the range, that is from 0 to $\tilde{u}_1$, and double the result to get ${}_nP_m$ since the integrand is symmetrical about the real axis. We then integrate along paths on which the real part x of u is constant, and the imaginary part y varies from y_1 to y_2. That is, we integrate from

$u = x + iy_1$ to $u = x + iy_2$, and the expression for ${}_nP_m$ is as follows:

$$ {}_nP_m = \frac{2(-1)^{n+m}}{\pi i} \int_{x+iy_1}^{x+iy_2} u\, \alpha_1^{-1} K^n \gamma^{-1}\, du \, . \qquad (5\text{-}1.2)$$

The limits of integration are as follows:

For $\tau < c_1^{-1}\, {}_nR_m$, $x = 0$, $y_1 = c_2^{-1}$, and

$y_2 = \text{Im}(\tilde{u}_1) = {}_nR_m^{-2}\, [\tau r - {}_nd_m(c_1^{-2}\, {}_nR_m^2 - \tau^2)^{1/2}]$.

For $\tau > c_1^{-1}\, {}_nR_m$, $x = \text{Re}(\tilde{u}_1) = {}_nR_m^{-2}\, {}_nd_m\, (\tau^2 - c_1^{-2}\, {}_nR_m^2)^{1/2}$,

$y_1 = 0$, and $y_2 = \text{Im}(\tilde{u}_1) = {}_nR_m^{-2}\, \tau r$.

5-2 COMPENSATED INTEGRAND

When evaluating the integral above numerically, care must be taken to avoid the singularity at $u = \tilde{u}_1$. One of the best ways of handling such a problem is the "compensated integrand" which we illustrate with the example below.

Denote the value of K at $u = \tilde{u}_1$ by K_1. We can then write

$$ {}_nP_m = I_1 + I_2 $$

$$ = \frac{2(-1)^{n+m}}{\pi i} \int_{x+iy_1}^{x+iy_2} u\alpha_1^{-1} (K^n - K_1^n)\, \gamma^{-1}\, du $$

$$ + \frac{2(-1)^{n+m}}{\pi i} K_1^n \int_{x+iy_1}^{x+iy_2} u\alpha_1^{-1}\, \gamma^{-1}\, du \, . \qquad (5\text{-}2.1)$$

The integrand of I_1 is then finite at all points of the integration path for $\tau \neq c_1^{-1}\, {}_nR_m$ since the limit as $u \to \tilde{u}_1$ is zero.

This integral can be reduced to integration over a real variable if we denote the integrand $P' = a + ib$ and the integration variable $u = x + iy$. Then I_1 can be written in real integrals as follows:

$$I_1 = \int_x^x a\,dx - \int_{y_1}^{y_2} b\,dy + i\int_{y_1}^{y_2} a\,dy + i\int_x^x b\,dx. \qquad (5\text{-}2.2)$$

The first and fourth integrals are zero since we are integrating along a path of constant real part x of u. The real contribution to I_1 is then the second integral, and we can write

$$I_1 = \frac{-2(-1)^{n+m}}{\pi} \int_{y_1}^{y_2} \text{Im}\left\{\frac{u\alpha_1^{-1}}{i}\,(K^n - K_1{}^n)\,\gamma^{-1}\right\} dy\;. \qquad (5\text{-}2.3)$$

Eliminating the "i", I_1 becomes

$$I_1 = \frac{2(-1)^{n+m}}{\pi} \int_{y_1}^{y_2} \text{Re}\left\{u\alpha_1^{-1}\,(K_1{}^n - K_1{}^n)\,\gamma^{-1}\right\} dy\;. \qquad (5\text{-}2.4)$$

The integral I_2 can be integrated analytically as shown below: The substitution $\omega = (c_1{}^{-2} + u^2)^{1/2}$ in I_2 of equation (5-2.1) yields

$$I_2 = \frac{2(-1)^{n+m}}{\pi i} K_1{}^n \int_{\omega_1}^{\omega_2} \left[{}_nR_m^2\,\omega^2 - 2\,{}_nd_m\,\tau\omega + (\tau^2 - c_1{}^{-2}r^2)\right]^{-1/2} d\omega, \quad (5\text{-}2.5)$$

where $\omega_1 = [c_1{}^{-2} + (x + iy_1)^2]^{1/2}$ and $\omega_2 = [c_1{}^{-2} + (x + iy_2)^2]^{1/2}$. The expression in brackets is of the form $1/\sqrt{X}$ where $X = a\omega^2 + b\omega + c$. The integrand is then the derivative of the function

$$F(\omega) = \frac{1}{\sqrt{a}} \log\left\{\sqrt{X} + \omega\sqrt{a} + \frac{b}{2\sqrt{a}}\right\}, \qquad (5\text{-}2.6)$$

or in expanded form,

$$F(\omega) = \frac{1}{{}_nR_m} \log \left\{ [{}_nR_m^2\, \omega^2 - 2\,{}_nd_m\, \tau\omega + (\tau^2 - c_1^{-2} r^2)]^{1/2} + \omega\, {}_nR_m - {}_nd_m\, \tau/{}_nR_m \right\}. \tag{5-2.7}$$

Thus we have

$$I_2 = \frac{2(-1)^{n+m}}{\pi i} \frac{K_1{}^n}{{}_nR_m} \log \left[\frac{f(\omega_2)}{f(\omega_1)} \right], \tag{5-2.8}$$

where

$$f(\omega) = [{}_nR_m \omega^2 - 2\,{}_nd_m\, \tau\omega + (\tau^2 - c_1^{-2} r^2)]^{1/2} + \omega\, {}_nR_m - {}_nd_m\, \tau/{}_nR_m . \tag{5-2.9}$$

Since $\gamma = 0$ at $u = \tilde{u}_1$, $X = 0$ at $\omega = \omega_2$, and I_2 can be written as

$$I_2 = \frac{2(-1)^{n+m}}{\pi i} \frac{K_1{}^n}{{}_nR_m} \log \left[\frac{{}_nR_m\, \omega_2 - {}_nd_m\, \tau/{}_nR_m}{f(\omega_1)} \right]. \tag{5-2.10}$$

Eliminating the "i", the final expression for the real contribution to I_2 is

$$I_2 = \frac{2(-1)^{n+m}}{\pi\, {}_nR_m} \operatorname{Im} \left\{ K_1{}^n \log \left[\frac{{}_nR_m\, \omega_2 - {}_nd_m\, \tau/{}_nR_m}{f(\omega_1)} \right] \right\}. \tag{5-2.11}$$

The expression for ${}_nP_m$ is then the sum of I_1, equation (5-2.4), and I_2, equation (5-2.11), as follows:

$$\begin{aligned} {}_nP_m = {} & \frac{2(-1)^{n+m}}{\pi} \int_{y_1}^{y_2} \operatorname{Re} \left\{ u\alpha_1^{-1} (K^n - K_1{}^n) \gamma^{-1} \right\} dy \\ & + \frac{2(-1)^{n+m}}{\pi\, {}_nR_m} \operatorname{Im} \left\{ K_1{}^n \log \left[\frac{{}_nR_m\, \omega_2 - {}_nd_m\, \tau/{}_nR_m}{f(\omega_1)} \right] \right\}, \end{aligned} \tag{5-2.12}$$

where $u = x + iy$.

We have avoided the singularity of ${}_nP_m'$ by setting the numerator zero when the denominator is zero to obtain a finite limit. Numerical calculations have shown that I_1' is a reasonably well behaved function so that I_1 can be evaluated using standard quadrature methods. Some difficulty is encountered in the immediate vicinity of the singularity at $\tau = c_1^{-1} R_m$, but I_1' is smooth enough to obtain sufficiently accurate results even in this range. The pressure ${}_nP_m$ can now be evaluated numerically with little difficulty using complex arithmetic.

REFERENCES

[1] Arons, A. B. and Yennie, D. R., "Phase Distortion of Acoustic Pulses Obliquely Reflected from a Medium of Higher Sound Velocity", Journal of the Acoustical Society of America, V. 22, 1950, pp. 231-237.

[2] Rosenbaum, J. H., "Shockwave Propagation in Shallow Water I", NAVORD Report 4353, 8 August 1956.

[3] Cagniard, L., Flinn, E. A., and Dix, C. H., Reflection and Refraction of Progressive Seismic Waves, McGraw-Hill Book Company, Inc., New York, 1962.

[4] Dix, C. H., "The Method of Cagniard in Seismic Pulse Problems", Geophysics, V. 19, 1954, pp. 727,728.

[5] Ewing, W. M., Jardetzky, W. S., and Press, F., Elastic Waves in Layered Media, McGraw-Hill Book Company, Inc., New York, 1957, pp. 13-14

[6] Ewing, Jardetzky, and Press (1957) pp. 131-132.

[7] Pekeris, C. L., "Theory of Propagation of Explosive Sound in Shallow Water", Geological Society of America Memoir 27, October 15, 1948, p. 47.

[8] Rosenbaum (1956), p. 6.

[9] Eichler, E., Rattayya, J. V., "Pressure Transmission from an Explosion in a Liquid Layer Overlying a Porous Solid, and Related Problems", Interim Technical Report U-184-134, Cambridge Acoustical Associates, Inc., Cambridge, Massachusetts, July 1964, pp. 15-22.

[10] Ewing, Jardetzky, and Press (1957) pp. 1-10.

[11] Spencer, T. W., "Reflection of an Acoustical Pressure Pulse from a Liquid-Solid Plane Boundary", Geophysics, V. 21, 1956, pp. 72-75.

[12] Roever, W. L., Vining, T. F., and Strick, E., 1959, "Propagation of Elastic Wave Motion from an Impulsive Source along a Fluid-Solid Interface", Phil. Trans. Roy. Soc. London, Ser. A, Vol. 251, pp. 455-523.

[13] Watson, G. N., A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 1958, pp. 17,18.

[14] Spencer (1956), pp. 84-86.

[15] Cagniard (1962), pp. 222,223,244,245.

[16] Scholte, J. G., 1948, "On the Large Displacements Commonly Regarded as Caused by Love Waves and Similar Dispersive Surface Waves", Konink. Ned. Akad. Wetenschap., Vol. 51, pp. 533-543, 642-649, 828-835, 969-976.

[17] Scholte, J. G., 1949, "On True and Pseudo Rayleigh Waves", Konink. Ned. Akad. Wetenschap., Vol. 52, pp. 652-653.

[18] Cagniard (1962), pp. 65,66.

[19] Townsend, E. J., Functions of a Complex Variable, Henry Holt and Company, New York, 1942, p. 286.

[20] Snay, H. G., "Bottom Reflection of Underwater Explosion Shock-waves: Elements and Military Significance", NOLTR 68-62, 11 June 1968, Confidential.

[21] Towne, D. H., "Pulse Shapes of Spherical Waves Reflected and Refracted at a plane Interface Separating two Homogeneous Fluids", Contribution No. 2044 Woods Hole Oceanographic Institution, 1967, pp.40-46.

[22] Cagniard (1962), pp. 60,61.

[23] Mackie, A. G., Boundary Value Problems, Hafner Publishing Company, New York, 1965, pp. 13,14.

[24] Churchill, R. V., Operational Mathematics, McGraw-Hill Book Company, Inc., New York, 1958, pp. 26,27.

APPENDIX A

MATHEMATICAL DETAILS IN THE INTEGRATION FOR THE STEP WAVE RESPONSE ${}_nP_m$

Many mathematical details in the integration of the step wave pressure response ${}_nP_m$ have been disregarded in the interest of arriving at a solution of the problem with as little complication as possible. But many of the mathematical details are essential for a good understanding of the problem and are thus the subject of this appendix.

A-1 CONCEPTS AND THEOREMS FROM COMPLEX ANALYSIS

Rosenbaum has used the "method of residues" to evaluate the complex integral for ${}_nP_m$. This procedure utilizes two important theorems from complex analysis: (a) Cauchy's theorem and (b) the residue theorem. A few preliminary concepts in the theory of complex variables will be discussed first.

The set of complex numbers is composed of numbers $z = x + iy$ where x and y are real numbers and $i^2 = -1$. Complex numbers are an extension of the real numbers and can be represented in a plane with x and y coordinates as shown in Figure A-1.1 below.

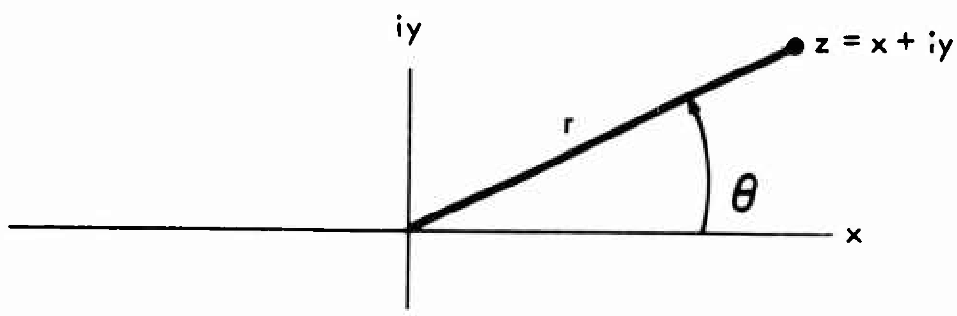

FIG. A-1.1 THE VARIABLE z IN THE COMPLEX PLANE

The complex number z can also be represented in polar coordinates (r, θ) where $r = |u| = (x^2 + y^2)^{1/2}$ is the absolute value, or modulus, of u and $\theta = \arctan(y/x)$ is the argument of u. The complex number z in polar coordinates is then

$$z = r \exp(i\theta) = r(\cos\theta + i \sin\theta). \qquad \text{(A-1.1)}$$

An important class of functions of a complex variable is the "analytic function". A function f(z) is termed "analytic" in the interior of a closed curve if that function is defined and has a derivative at each point inside the curve. The function must also be single-valued in the area of definition, that is, for each z the function f(z) must have one and only one value.

Those points at which the function is not analytic are termed "singularities". If the singularity can be enclosed by a circle, however small, which does not include other singularities, it is an "isolated" singularity. Singularities such as those on the "branch line" of a complex square root are not isolated.

We are now ready to state the two theorems used in the "method of residues". Cauchy's theorem states that if a function f(z) is analytic inside and on a closed integration path C, then we have

$$\int_C f(z)\, dz = 0. \qquad \text{(A-1.2)}$$

The residue theorem states that the integral

$$\int_C f(z)\, dz$$

around the closed path C, which encloses only isolated singularities, is equal to $2\pi i$ times the sum of the "residues" of the enclosed singularities. The counterclockwise direction along the path C is taken as positive. If the function f(z) has N singularities enclosed by C, then we obtain

$$\int_C f(z)\, dz = 2\pi i \sum_{j=1}^{N} Rj \ , \qquad \text{(A-1.3)}$$

where Rj is the "residue" of f(z) at the jth singularity.

The "residue" of a function f(z) at an isolated singularity z_0 can be defined as the coefficient a_1 in the power series expansion

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n}{(z - z_0)^n} + b_0 + \sum_{n=1}^{\infty} b_n (z - z_0)^n \ . \qquad \text{(A-1.4)}$$

This expansion is called a "Laurent's series of f(z) about z_0".

In applying the above theorems for the evaluation of ${}_nP_m$, we must examine the functions in the integrand ${}_nP_m'$ and the location and nature of their singularities. The integrand ${}_nP_m'$ contains square roots which are multiple-valued functions in the complex plane. We must then define single-valued functions in the region of integration. To illustrate the procedure for defining a single-valued function from a multiple-valued one, we will study the square root $z^{1/2}$ as a simple example of a bi-valued function.

A-2 STUDY OF THE SQUARE ROOT $w = z^{1/2}$

The complex square root $w = z^{1/2}$ is defined in polar coordinates as

$$z^{1/2} = r^{1/2}\exp(\tfrac{\theta}{2}) = r^{1/2}\left[\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}\right] . \qquad \text{(A-2.1)}$$

If θ in equation (A-2.1) is increased by 2π, $z^{1/2}$ assumes the negative of its original value. To obtain all possible values of $z^{1/2}$, the argument θ must be increased by 4π, that is, the vector shown in Figure A-2.1 must make two full circuits in the z-plane. Hence, the square root $z^{1/2}$ takes on two values at every point in the z-plane.

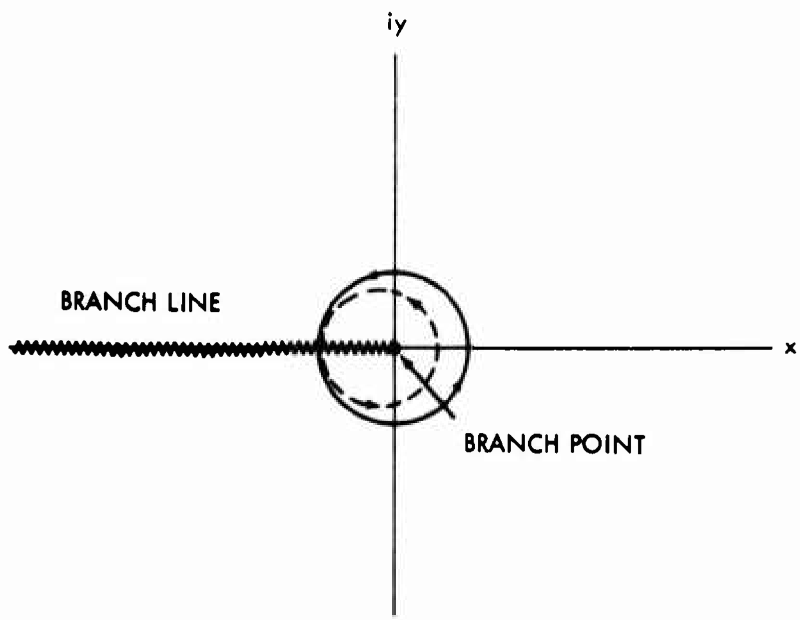

FIG. A-2.1 EXTENDED z-PLANE FOR $z^{1/2}$

The concept of the Riemann surface allows for a representation of such multiple-valued functions. For $z^{1/2}$ the Riemann surface consists of two sheets, R_1 and R_2, one lying above the other and each containing a single value of $z^{1/2}$ as shown in Figure A-2.2.

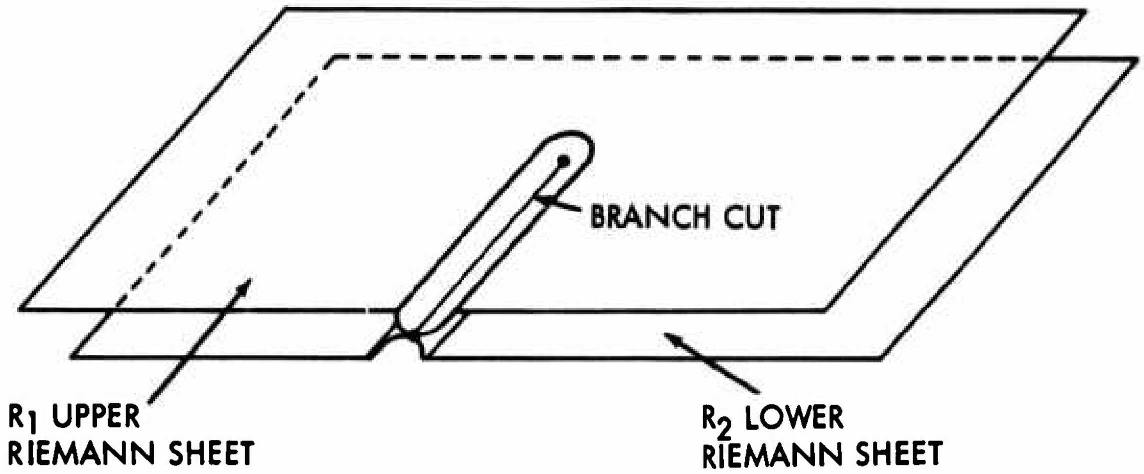

FIG. A-2.2 RIEMANN SURFACE FOR $z^{1/2}$

When the point z makes the first full circuit, say from $-\pi$ to π, the argument $\phi = \theta/2$ has changed from $-\pi/2$ to $+\pi/2$ and the values of $z^{1/2}$ are obtained from values of z in the first or top sheet R_1. For the second circuit from $\theta = \pi/2$ to 3π, the values of the square root $z^{1/2}$ are obtained from values of z in the second or lower Riemann sheet R_2.

The transition from the top sheet to the lower sheet takes place at the "branch cut" or "branch line" where the two sheets cross each other so that R_1 merges into R_2 and R_2 merges into R_1, as illustrated in Figure A-2.3 below.

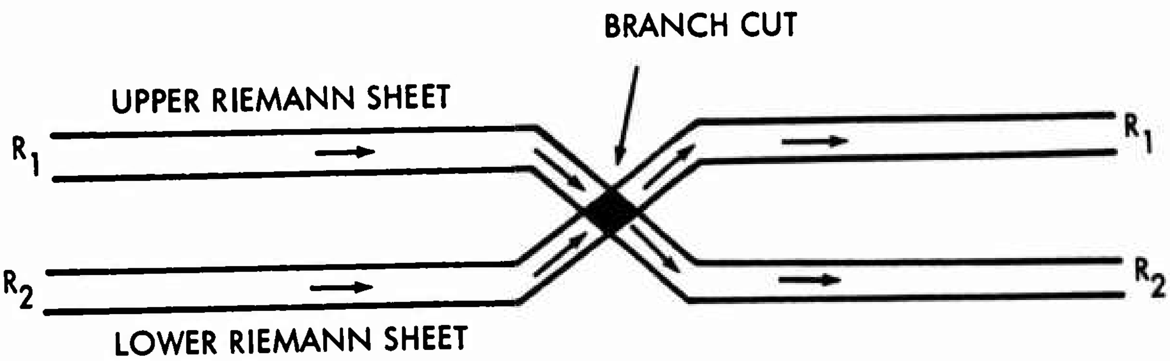

FIG. A-2.3 CROSSING OF THE RIEMANN SHEETS AT THE BRANCH LINE

At the end of the second circuit the value of $z^{1/2}$ moves from the lower sheet R_2 back into the upper sheet R_1. By this way an analytic, i.e. a single-valued, function of $z^{1/2}$ is defined in each sheet except for places where the sheets cross each other namely on the branch cut.

For the purpose of integrations, the path of integration and the branch line can be chosen in such a way that a crossing of the branch line can be avoided. The function then remains analytic. The negative real axis is usually taken as the branch cut, even though any ray originating at $z = 0$ may be used. The choice of the ray depends on the particular problem under consideration. However, the end point of the branch cut ($z = 0$ in our case) is of special significance and cannot be chosen arbitrarily. The crossing of the two Riemann sheets at the branch cut ends at the point $z = 0$. This point is called the "branch point" and it is common to both sheets. That means that the roots of z must have the same value. In our case $z^{1/2} = 0$ is a common root and constitutes a branch point. The other branch point is $z^{1/2} = \infty$ and it is seen that the branch line connects these two branch points.

A-3 BRANCH CUTS FOR THE MULTIPLE-VALUED FUNCTIONS IN THE INTEGRAND OF ${}_nP_m$

Before the integral for the step wave pressure response ${}_nP_m$ can be evaluated using Cauchy's theorem, branch cuts must be made for the multiple-valued functions in the integrand. Branch cuts must be made for $\alpha_i = (u^2 + c_i^{-2})^{1/2}$ for $i = 1, 2, 3, 4$, and $\gamma = \left\{u^2r^2 + [\tau - d_m(c_1^{-2} + u^2)^{1/2}]^2\right\}^{1/2}$. The function α_2 is contained in the liquid bottom reflection coefficient K, and the functions α_3 and α_4 are contained in the rigid bottom K.

Other singularities of the integrand which must be considered are the zeroes of the denominator of K and the singularity at infinity. The derivative of the integrand ${}_nP_m'$ at infinity is not defined. The nature of this singularity at infinity will be further examined when we study the residues of the integrand. The singularities arising from the zeroes of the denominator of K are treated below.

THE SQUARE ROOTS α_i

First we examine the square roots α_i. Since the square roots for i = 1, 2, 3, and 4 are of the same form, we will examine only $\alpha_1 = (u^2 + c_1^{-2})^{1/2}$. Factoring α_1 into $(u + ic_1^{-1})^{1/2}(u - ic_1^{-1})^{1/2}$, it is clear that α_1 is the product of two functions which have branch points at $u = -ic_1^{-1}$ and $u = ic_1^{-1}$ and at infinity. In the following discussion we show that when these two factors are combined the branch cut can be made from $-ic_1^{-1}$ to ic_1^{-1} along the imaginary axis. For this purpose we set $u + ic_1^{-1} = r_1 \exp(i\theta_1)$ and $u - ic_1^{-1} = r_2 \exp(i\theta_2)$ as shown in Figure A-3.1. Then α_1 becomes

$$\begin{aligned}\alpha_1 &= [r_1 \exp(i\theta_1)]^{1/2}[r_2 \exp(i\theta_2)]^{1/2} \\ &= \sqrt{r_1 r_2}\, \exp[\tfrac{i}{2}(\theta_1 + \theta_2)] \\ &= \sqrt{r_1 r_2}\, [\cos(\tfrac{\theta_1 + \theta_2}{2}) + i \sin(\tfrac{\theta_1 + \theta_2}{2})]. \qquad \text{(A-3.1)}\end{aligned}$$

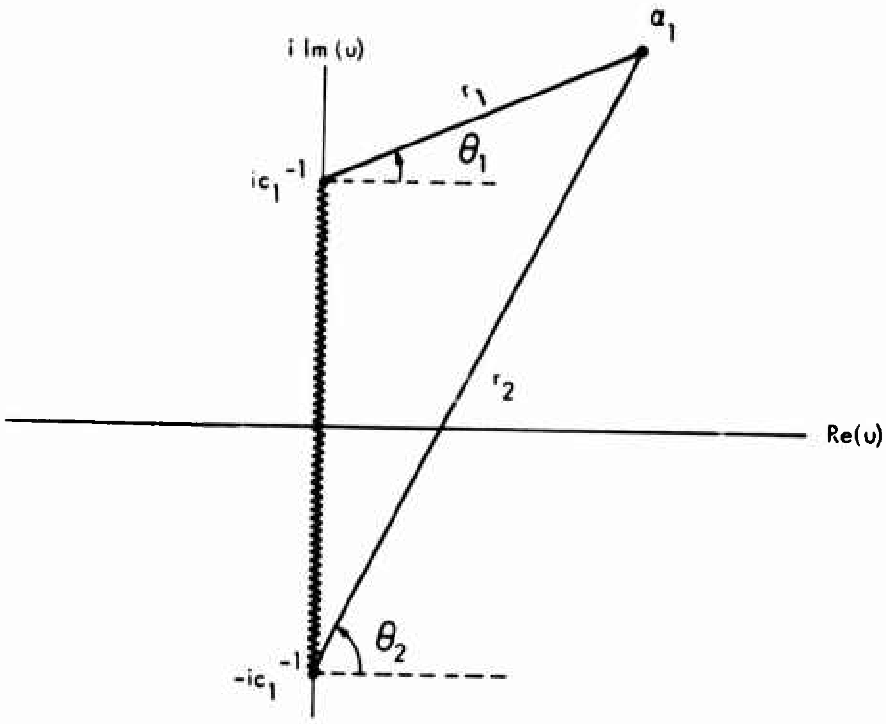

FIG. A-3.1 THE SQUARE ROOT $\alpha_1 = (u^2 + c_1^{-2})^{1/2}$

For the square root $[r_1 \exp(i\theta_1)]^{1/2}$ make the branch cut along the ray $\theta_1 = -\pi/2$ from ic_1^{-1} to infinity along the negative imaginary axis, and for $[r_2 \exp(i\theta_2)]^{1/2}$ make a branch cut along the ray $\theta_2=-\pi/2$ from $-ic_1^{-1}$ to infinity along the negative imaginary axis. Hence we define the principal branch (first or top sheet of the Riemann plane) of α_1 as follows:

α_1 is defined by equation (A-3.1) and is restricted to the domain $-\frac{\pi}{2} \le \theta_1 < \frac{3\pi}{2}$, $-\frac{\pi}{2} \le \theta_2 < \frac{3\pi}{2}$, $r_1 > 0$, $r_2 > 0$, $r_1 + r_2 > 2c_1^{-1}$. That this branch defines an analytic function except along the cut from $-ic_1^{-1}$ to ic_1^{-1} will now be shown.

The square root α_1 is analytic along the positive imaginary axis from ic_1^{-1} to infinity since no branch line is crossed. Along the negative imaginary axis from $-ic_1^{-1}$ to infinity, α_1 is analytic since

both $[r_1 \exp(i\theta_1)]^{1/2}$ and $[r_2 \exp(i\theta_2)]^{1/2}$ change sign from $\sqrt{r_1}$ to $-\sqrt{r_1}$ and from $\sqrt{r_2}$ to $-\sqrt{r_2}$ as the point crosses the axis. The two sign changes cancel leaving α_1 continuous. Along the interval $-ic_1^{-1}$ to ic_1^{-1}, $[r_1 \exp(i\theta_1)]^{1/2}$ jumps from $\sqrt{r_1}$ to $-\sqrt{r_1}$ but $[r_2 \exp(i\theta_2)]^{1/2}$ is continuous as the point z crosses the interval. Thus α_1 is discontinuous as z crosses this interval and not analytic on it. The square root α_1 is then analytic everywhere except along the line from $-ic_1^{-1}$ to ic_1^{-1} which represents the branch cut.

In a similar manner analytic branches of α_i, i = 2, 3, 4 can be defined with branch cuts from $-ic_i^{-1}$ to ic_i^{-1} along the imaginary axis. Then using equation (A-3.1) and its α_i counterparts one can show that on the top sheets of the Riemann planes for α_i the square roots have (1) positive real parts in the right half of the u-plane, (2) negative real parts in the left half plane, (3) positive imaginary parts in the upper half plane and, (4) negative imaginary parts in the lower half plane.

A-4 BRANCH POINTS OF FUNCTION $\gamma = \left\{u^2r^2 + [\tau - d_m\alpha_1]^2\right\}^{1/2}$

The branch points of γ are the roots of $\gamma^2 = 0$. The term γ^2 can be factored into

$$\gamma^2 = (\tau - \alpha_1 d_m - iur)(\tau - \alpha_1 d_m + iur). \qquad \text{(A-4.1)}$$

The solutions are then

$$u = R_m^{-2}[\pm i\tau r \pm d_m(\tau^2 - c_1^{-2}R_m^{-2})^{1/2}]. \qquad \text{(A-4.2)}$$

With the substitutions $\cos \ell_m = d_m/R_m$, $\sin \ell_m = r/R_m$, and $\tau_m = \tau/R_m$ the solutions can be written

$$u = \pm\, i\, \tau_m \sin \ell_m \pm (\tau_m^2 - c_1^{-2})^{1/2} \cos \ell_m. \tag{A-4.3}$$

Thus γ has four branch points compared to only two for α_1. But branch cuts can be made for γ as for α_1 between branch points which are complex conjugates. Two cuts are then necessary, but because of the expression $(\tau_m^2 - c_1^{-2})^{1/2}$, different cuts are required depending on whether $\tau_m < c_1^{-1}$ or $\tau_m > c_1^{-1}$. These two cases must be discussed separately.

A-5 BRANCH CUTS FOR γ WHEN $\tau_m < c_1^{-1}$

When $\tau_m < c_1^{-1}$ the four zeroes of γ may be written

$$u_1 = i\,[-\tau_m \sin \ell_m + (c_1^{-2} - \tau_m^2)^{1/2} \cos \ell_m] \tag{A-5.1}$$

$$\tilde{u}_1 = i\,[\tau_m \sin \ell_m - (c_1^{-2} - \tau_m^2)^{1/2} \cos \ell_m] \tag{A-5.2}$$

$$u_3 = i\,[-\tau_m \sin \ell_m - (c_1^{-2} - \tau_m^2)^{1/2} \cos \ell_m] \tag{A-5.3}$$

$$\tilde{u}_3 = i\,[\tau_m \sin \ell_m + (c_1^{-2} - \tau_m^2)^{1/2} \cos \ell_m] \tag{A-5.4}$$

The branch cuts are then made along the imaginary axis from u_1 to $\tilde{u}_1$ and from u_3 to $\tilde{u}_3$. Since $\tau > c_1^{-1} d_m$, or $\tau_m > c_1^{-1} \cos \ell_m$, u_1 and u_3 are below the real axis, and $\tilde{u}_1$ and $\tilde{u}_3$ are above the real axis. In order that γ be positive for positive, real u, we define γ to have a positive real part right of the cut.

From equations (A-5.3) and (A-5.4) we see that u_3 and $\tilde{u}_3$ are beyond the integration path from u_1 and $\tilde{u}_1$ on the imaginary axis and cause no problem in the integral for ${}_nP_m$ when $\tau_m < c_1^{-1}$.

Before we can carry out the integration in this time range using real variables, we must know whether γ is real or imaginary between u_1 and $\tilde{u}_1$. Consequently, we must also know whether $|u_1| < c_1^{-1}$ or $|u_1| > c_1^{-1}$ for $\tau_m < c_1^{-1}$ since $\alpha_1 = (u^2 + c_1^{-2})^{1/2}$ is contained in γ.

A-6 γ IS REAL AND $|u_1| < c_1^{-1}$ FOR $\tau_m < c_1^{-1}$

For $\tau_m < c_1^{-1}$ the absolute value $|u_1| = \tau_m \sin \ell_m - (c_1^{-2} - \tau_m^2)^{1/2} \cos \ell_m$, which increases as τ_m increases, reaches a maximum at $\tau_m = c_1^{-1}$. The maximum $|u_1| = c_1^{-1} \sin \ell_m$ is less than or equal to c_1^{-1}. Thus $|u_1|$ is always less than c_1^{-1} for $\tau_m < c_1^{-1}$.

In order to show that γ is real for $|u_1| < c_1^{-1}$, make the change of variables $u = ix$ and $\omega = (c_1^{-2} - x^2)^{1/2}$. The following expressions for the zeroes of γ are then obtained:

$$\omega_1 = \tau_m \cos \ell_m + (c_1^{-2} - \tau_m^2)^{1/2} \sin \ell_m \qquad \text{(A-6.1)}$$

$$\omega_2 = \tau_m \cos \ell_m - (c_1^{-2} - \tau_m^2)^{1/2} \sin \ell_m \; . \qquad \text{(A-6.2)}$$

Thus γ is real when $(\omega - \omega_1)(\omega - \omega_2) \geq 0$, or equivalently, when $\omega \geq \omega_1$ or $\omega \leq \omega_2$. It can be shown that u_1 and $\tilde{u}_1$ correspond to $\omega = \omega_1$. Since $|x| \leq |u_1| < c_1^{-1}$ on the integration path for $\tau_m < c_1^{-1}$, it follows that $\omega \geq \omega_1$. Therefore, γ is real in the time range $\tau_m < c_1^{-1}$.

A-7 BRANCH POINTS OF γ WHEN $\tau_m > c_1^{-1}$

When $\tau_m > c_1^{-1}$ the zeroes of γ^2 are as follows:

$$u_1 = -i\tau_m \sin \ell_m + (\tau_m^2 - c_1^{-2})^{1/2} \cos \ell_m , \quad (A\text{-}7.1)$$

$$\tilde{u}_1 = i\tau_m \sin \ell_m + (\tau_m^2 - c_1^{-2})^{1/2} \cos \ell_m , \quad (A\text{-}7.2)$$

$$u_3 = -i\tau_m \sin \ell_m - (\tau_m^2 - c_1^{-2})^{1/2} \cos \ell_m , \quad (A\text{-}7.3)$$

$$\tilde{u}_3 = i\tau_m \sin \ell_m - (\tau_m^2 - c_1^{-2})^{1/2} \cos \ell_m , \quad (A\text{-}7.4)$$

Since $\tau_m \sin \ell_m$ is positive, u_1 and $\tilde{u}_1$ are respectively in the fourth and first quadrants, and u_3 and $\tilde{u}_3$ are respectively in the third and second quadrants. The branch cuts are then made from u_1 to $\tilde{u}_1$ and from u_3 to $\tilde{u}_3$. In order that the real part of γ be positive on the integration paths which are left of the cut from u_1 to $\tilde{u}_1$, we define γ to have a positive real part left of the cut from u_1 to $\tilde{u}_1$ and a negative real part right of the cut.

A-8 THE BRANCH POINTS OF γ ON THE TOP RIEMANN SHEET FOR $\tau_m > c_1^{-1}$

To evaluate the integral for ${}_nP_m$ for $\tau_m > c_1^{-1}$ by the "method of residues", we must know which branch points of γ are on the top sheet of the Riemann plane, that is, which branch points contribute to the integral.

From equation (A-4.1) we may obtain the following solutions for τ as a function of u when γ is zero:

$$\tau = \alpha_1 d_m + iur \quad (A\text{-}8.1)$$

$$\tau = \alpha_1 d_m - iur , \quad (A\text{-}8.2)$$

where $\alpha_1 = (u^2 + c_1^{-2})^{1/2}$.

Since the integration in u is to be carried out on the Riemann sheet which corresponds to positive, real time τ, we will show that the branch points of γ on the top sheet are the limits of integration u_1 and $\tilde{u}_1$. This can be seen by considering the following paths for u shown in Figure A-8.1:

1. The counterclockwise infinite semicircle ABCDEA
2. The counterclockwise infinite semicircle FGHIJF

The paths along the imaginary axis are allowed to become arbitrarily close to the axis and extend from $-\infty$ to $+\infty$. The points J and E approach, respectively, $-\infty$ and $+\infty$ along the real axis.

As u traverses the above paths, τ from equations (A-8.1) and (A-8.2) traverses, respectively, the paths in Figure A-8.2 and Figure A-8.3 below. The primed letters represent the image in the τ-plane of the corresponding unprimed letters in the u-plane. Between B′ and C′ and between H′ and G′, τ is real. On the other parts of the paths τ is complex. The shaded region enclosed by O′C′D′E′O′ in the τ-plane of Figure A-8.2 is the image of the fourth quadrant of the u-plane. Similarly, the shaded region enclosed by O′E′A′B′O′ in the τ-plane of Figure A-8.3 is the image of the first quadrant of the u-plane. Thus for real $\tau > c_1^{-1}d_m$ the zeroes of γ must be in the right half of the u-plane or along the imaginary axis. Consequently, the only branch points on the top sheet of the Riemann plane when $\tau > c_1^{-1}R_m$ or $\tau_m > c_1^{-1}$ are the integration limits u_1 and $\tilde{u}_1$.*

*The general method used here to study the branch points of γ is that used by Cagniard [18] pp 65,66 to study his corresponding branch points for the problem in which both media are solids.

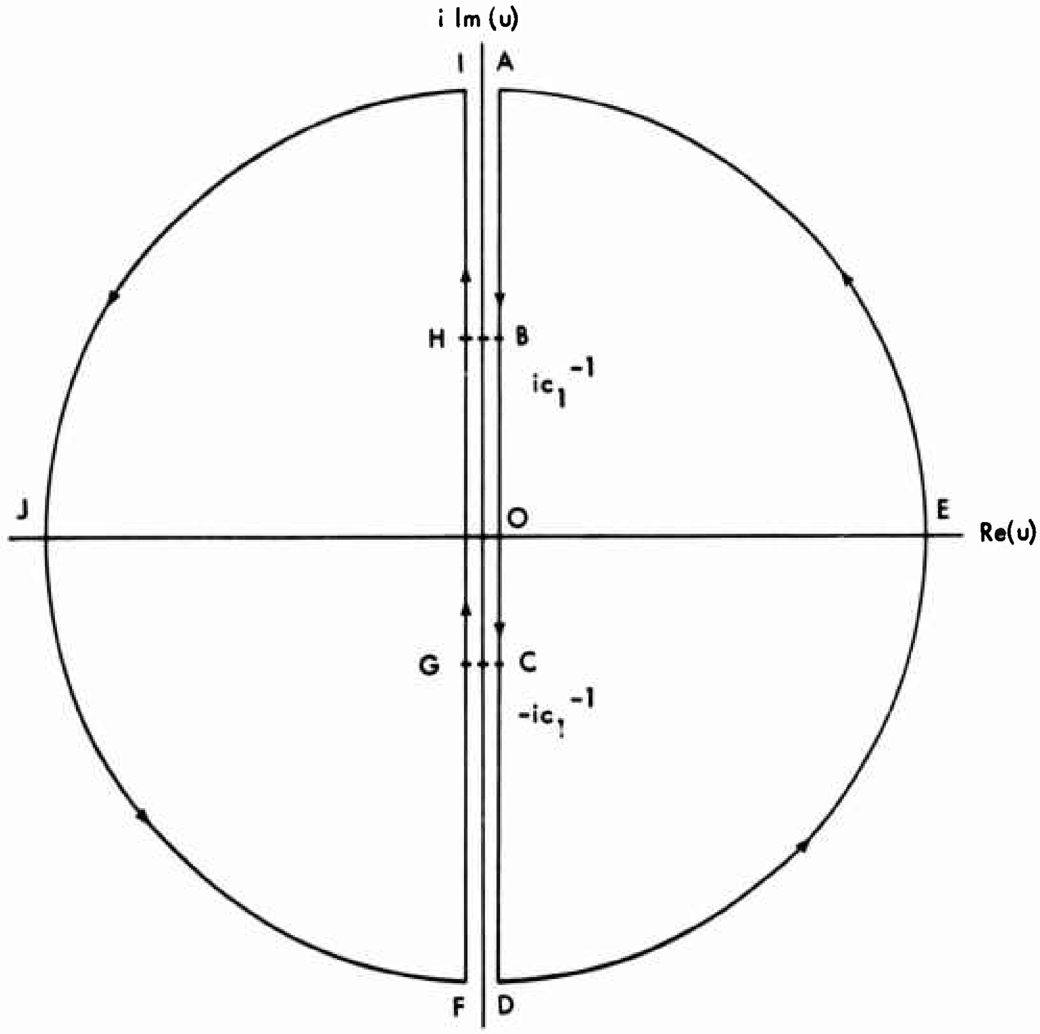

FIG. A-8.1 CONTOURS OF THE POINT υ IN THE υ-PLANE

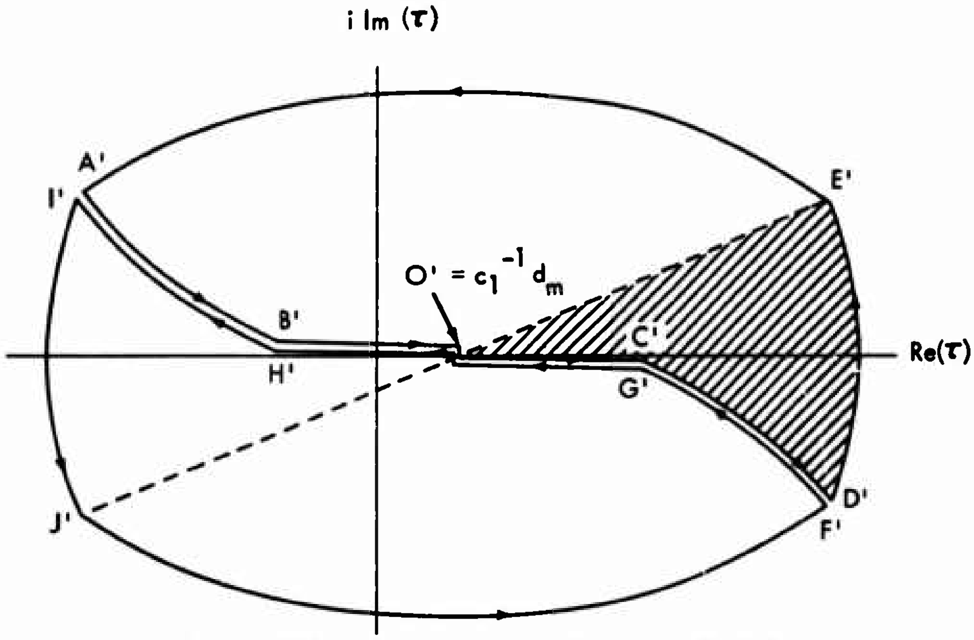

FIG. A-8.2 PATH OF $\tau = \alpha_1 d_m + iur$ IN THE τ-PLANE

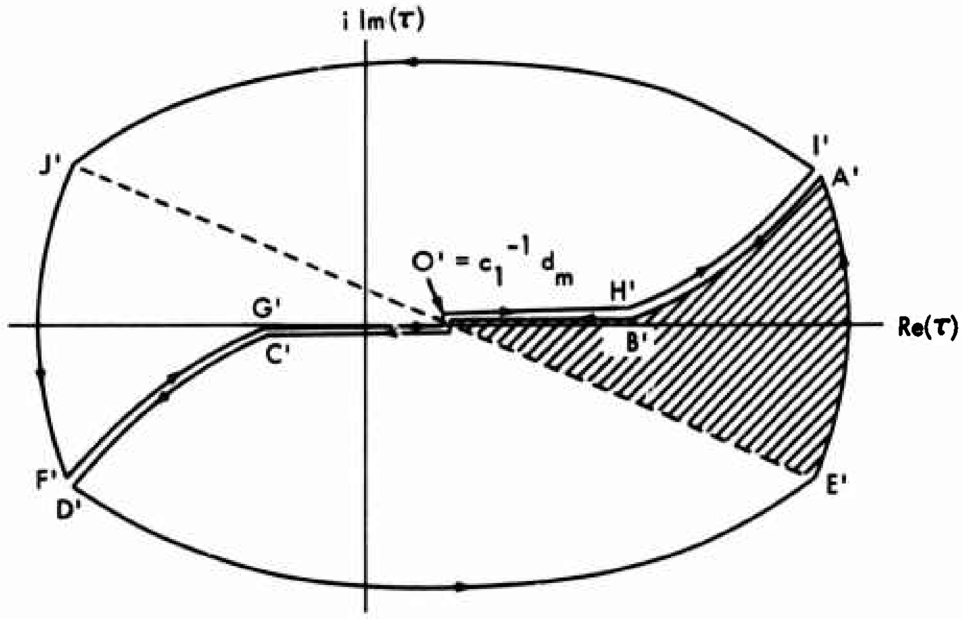

FIG. A-8.3 PATH OF $\tau = \alpha_1 d_m - iur$ IN THE τ-PLANE

A-9 γ IS REAL FOR $\tau_m > c_1^{-1}$ WHEN $|u| \le c_1^{-1}$

In the integration of ${}_nP_m$ for $\tau_m > c_1^{-1}$ by the "method of residues" we must know the range over which γ is real along the imaginary axis. Since α_1 is imaginary for $|u| > c_1^{-1}$ making γ complex, the range $|u| \le c_1^{-1}$ offers the only possibility for real γ.

Therefore consider γ along the imaginary axis with $\tau_m > c_1^{-1}$ and $|u| \le c_1^{-1}$. The change of variables $u = ix$ in equation (A-4.1) requires that for γ to be real we must have

$$\gamma^2 = (\tau - \alpha_1 d_m + xr)(\tau - \alpha_1 d_m - xr) > 0, \qquad \text{(A-9.1)}$$

and hence

$$\begin{gathered} \tau - \alpha_1 d_m + xr > 0 \text{ and } \tau - \alpha_1 d_m - xr > 0 \\ \text{or} \\ \tau - \alpha_1 d_m + xr < 0 \text{ and } \tau - \alpha_1 d_m - xr < 0 \,. \end{gathered} \qquad \text{(A-9.2)}$$

For $|x| > 0$ and $|x|r > 0$ the appropriate condition on τ is then

$$\tau - \alpha_1 d_m - |x|r > 0 \,, \qquad \text{(A-9.3)}$$

or equivalently,

$$\tau > (c_1^{-2} - x^2)^{1/2} d_m + |x|r \quad . \qquad \text{(A-9.4)}$$

The maximum value of $\alpha_1 d_m + |x|r$ for $|x| < c_1^{-1}$ is $c_1^{-1} R_m$ which

occurs when $|x| = c_1^{-1} r/R_m < c_1^{-1}$. Hence γ is real along the imaginary axis for $\tau > c_1^{-1}R_m$ or $\tau_m > c_1^{-1}$ and $|x| < c_1^{-1}$.

This completes the study of the multiple-valued functions and their singularities in ${}_nP_m'$. We now treat the singularities of ${}_nP_m'$ at the zeroes of the denominator of the reflection coefficient K.

A-10 SINGULARITIES IN THE REFLECTION COEFFICIENT K

The reflection coefficient K for liquid and rigid bottoms are quite different and are to be treated separately. First, the reflection coefficient for a liquid bottom equation (1-7.8) is

$$K = \frac{\alpha_1 - b\alpha_2}{\alpha_1 + b\alpha_2}. \qquad \text{(A-10.1)}$$

The singularity occurs if $\alpha_1 + b\alpha_2 = 0$, that is if $\alpha_1 = -b\alpha_2$. But from the above discussion we have seen that for a given value of u the real and imaginary parts of α_1 and α_2 have the same sign on the same sheet of the Riemann plane; hence, $\alpha_1 = -b\alpha_2$ is impossible if α_1 and α_2 are restricted to the same Riemann sheets. Thus a liquid bottom K has no singularities on the top sheet in which the integration is performed. We therefore conclude that the integrand of ${}_nP_m$ for a liquid bottom is analytic everywhere except on the branch cuts and at infinity.

The reflection coefficient for a rigid bottom is

$$K = \frac{\alpha_1[(2u^2 + c_4^{-2})^2 - 4u^2\alpha_3\alpha_4] - b\alpha_2\ c_4^{-4}}{\alpha_1[(2u^2 + c_4^{-2})^2 - 4u^2\alpha_3\alpha_4] + b\alpha_2\ c_4^{-4}}. \qquad \text{(A-10.2)}$$

The denominator of K has zeroes $\pm ik$ called Stonley poles which can

be found numerically. It has been shown by Spencer [14] that the Stonley poles are beyond the branch points $\pm ic_1^{-1}$, $\pm ic_3^{-1}$, and $\pm ic_4^{-1}$ on the imaginary axis of the u-plane and also that $\pm ik$ are the only poles on the top sheet of the Riemann plane. Further references to the Stonley poles are made in Section 3-7 of the text.

We have now completed the discussion of the singularities in the integrand of ${}_nP_m$. The integration can be performed if the paths are chosen to avoid the branch cuts. In the next section an appropriate integration path is discussed which can be used to reduce the complex integral to a real integral.

A-11 THE METHOD OF RESIDUES IN EVALUATING INTEGRALS

The application of the "method of residue" to evaluate an integral in the complex plane can be best explained by an example. The problem of evaluating the rigid bottom integral for ${}_nP_m$ when $\tau > c_1^{-1}\,{}_nR_m$ is appropriate. The integration paths used are shown in figure A-11.1 below. The original integration path is shown as the curve $u_1A\tilde{u}_1$. Denote the integrand of ${}_nP_m$ as ${}_nP_m'$. From equation (1-11.16) the integrand is

$$ {}_nP_m' = \frac{(-1)^{n+m}}{\pi i}\,\frac{uK^n}{\alpha_1\gamma}\ . \qquad \text{(A-11.1)}$$

Since u, α_1, α_3, α_4, and K^n are the same on path $u_1A\tilde{u}_1$ as on $u_1B\tilde{u}_1$ and since γ has opposite signs on these paths, we can write

$$ {}_nP_m = \frac{1}{2}\oint_{\theta}\,{}_nP_m'\,du. \qquad \text{(A-11.2)}$$

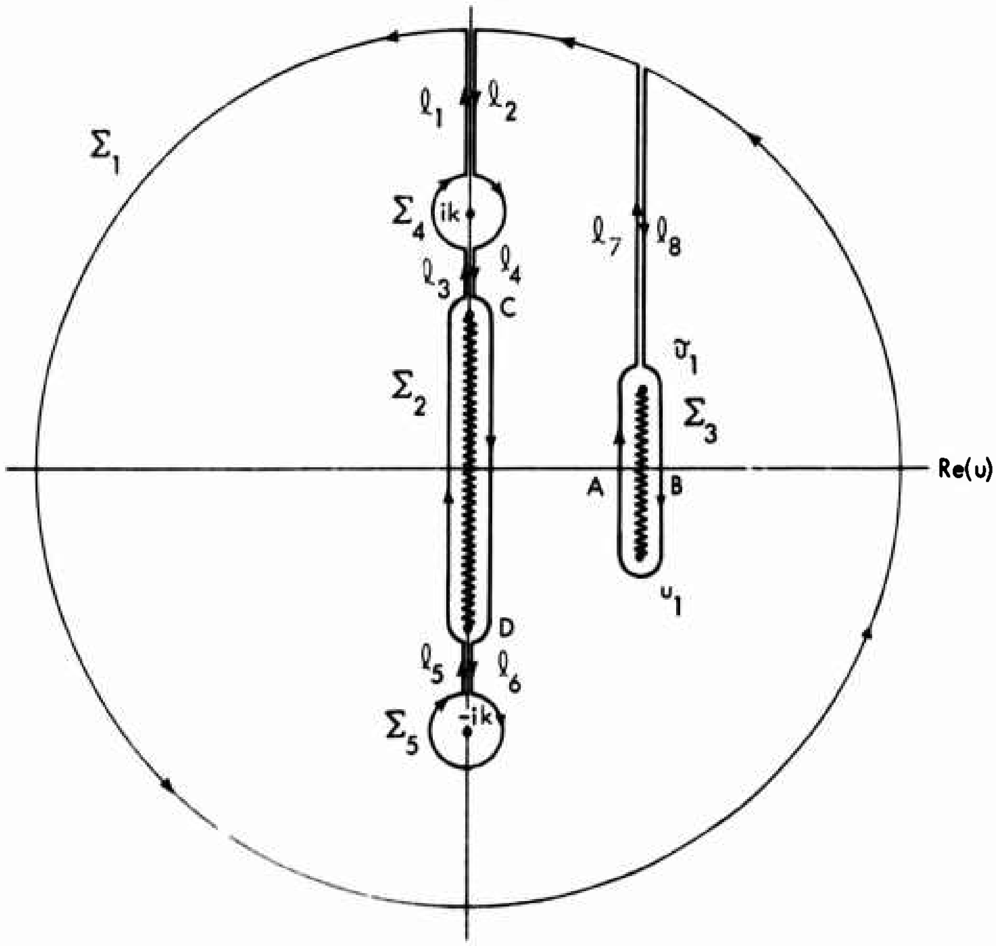

FIG. A-11.1 PATH OF INTEGRATION IN THE υ-PLANE FOR A RIGID BOTTOM

The Σ_3 and similar symbols placed on the integral sign denote the path of integration.

The integration path shown in Figure A-11.1 above is composed of the following parts: the branch line paths Σ_2 and Σ_3, the small circles Σ_4 ane Σ_5 around the Stonley poles, the paths ℓ_j for $j = 1 - 8$, and the large circle Σ_1 which encloses the other paths. This total path does not include any singularities in its interior and hence by Cauchy's theorem the integral of ${}_nP'_m$ over this path is zero.

In the limiting process the paths ℓ_1 and ℓ_2, ℓ_3 and ℓ_4, etc., approach the same path and hence the integrals along these paths cancel in pairs. Paths Σ_1, Σ_4 and Σ_5 then become closed circles. In paths Σ_2 and Σ_3 the vertical parts are allowed to become arbitrarily close to the branch cuts but do not approach the same path as with the ℓ_j because the multiple-valued functions change signs discontinuously as the branch lines are crossed. The curved end paths of Σ_2 and Σ_3 reduce to the points C, D, A, and B at the end of the vertical paths. It can be shown that the integrals are convergent on these paths and hence can be carried between the branch points.

The integration over the above path then reduces to the following:

$$\oint_{\Sigma_1} {}_nP'_m \, du + \oint_{\Sigma_2} P' du + \oint_{\Sigma_3} P' du + \oint_{\Sigma_4} P' du + \oint_{\Sigma_5} P' du = 0. \qquad \text{(A-11.3)}$$

Hence we can write

$$ {}_nP_m = \int_{u_1}^{\tilde{u}_1} {}_nP'_m \, du = \frac{1}{2} \oint_{\Sigma_3} P' du = -\frac{1}{2}\left[\oint_{\Sigma_1} P' du + \oint_{\Sigma_2} P' du + \oint_{\Sigma_4} P' du + \oint_{\Sigma_5} P' du\right]. \qquad \text{(A-11.4)}$$

Path Σ_1 is allowed to have an infinite radius and is said to "enclose" the singularity at infinity. Similarly, paths Σ_4 and Σ_8 enclose the Stonley poles $\pm ik$. Applying the residue theorem, we obtain

$$ {}_nP_m = -\pi i \left[R_\infty - (R_{ik} + R_{-ik}) \right] - \frac{1}{2} \oint_{\Sigma_2} {}_nP_m' \, du, \qquad \text{(A-11.5)} $$

where R_∞, R_{ik}, and R_{-ik} are the residues of ${}_nP_m'$ at ∞, ik, and -ik respectively. The negative sign immediately before $(R_{ik} + R_{-ik})$ appears because the contours Σ_4 and Σ_8 are taken in the negative (clockwise) direction.

Using the method of residues, ${}_nP_m$ can then be evaluated when the residues R_∞, R_{ik}, and R_{-ik} and the branch line integral on Σ_2 are known. For a liquid bottom the Stonley poles do not exist so the problem reduces to finding R_∞ and the integral on Σ_2. The residues for both liquid and rigid bottoms are derived in the next section. The branch line integral for a liquid bottom is discussed in Chapter II, and the branch line integral for a rigid bottom is discussed in Chapter IV. These integrals are complicated and have different expressions depending on the relative sizes of the velocities c_1 and c_2 in a liquid bottom and c_1, c_3, and c_4 in a rigid bottom.

A-12 RESIDUE AT INFINITY

To obtain the residue R_∞ of ${}_nP_m$ at infinity the integrand of equation (1-11.14), namely

$$ {}_nP_m' = \frac{(-1)^{n+m}}{\pi i} \frac{uK^n}{\alpha_1 [u^2 r^2 + (\tau - d_m \alpha_1)^2]^{1/2}}, \qquad \text{(A-12.1)} $$

must be expanded in a Laurent's series about $u = \infty$. After dividing numerator and denominator of equation (A-12.1) by u^2 and substituting $z = 1/u$, the integrand becomes

$$ {}_nP_m'(z) = \frac{(-1)^{n+m}z}{\pi i}\left\{\frac{[K(z)]^n}{(1+z^2c_1^{-2})^{1/2}\left\{r^2+[\tau z-d_m(1+z^2c_1^{-2})^{1/2}]^2\right\}^{1/2}}\right\}. \tag{A-12.2} $$

Expansion of each expression in a Taylor's series about $z = 0$ yields

$$ {}_nP_m'(z) = \frac{(-1)^{n+m}z}{\pi i}\,\frac{[K(z=0)+\sum_{j=1}^{\infty} a_j\, z^j]^n}{[1+\sum_{j=1}^{\infty} b_j\, z^j][R_m+\sum_{j=1}^{\infty} c_j\, z^j]}\,. \tag{A-12.3} $$

Multiplying the two series in the denominator and dividing their product into the series for K^n, we obtain

$$ {}_nP_m'(z) = \frac{(-1)^{n+m}z}{\pi i}\left\{\frac{1}{R_m}[K(z=0)]^n + \sum_{j=1}^{\infty} A_j\, z^j\right\}, \tag{A-12.4} $$

or

$$ {}_nP_m' = \frac{(-1)^{n+m}}{\pi i}\left\{\frac{[K(\infty)]^n/R_m}{u} + \sum_{j=2}^{\infty}\frac{A_j}{u^j}\right\}. \tag{A-12.5} $$

The residue is then*

$$ R_\infty = -\left\{\frac{(-1)^{n+m}}{\pi i\, R_m}[K(\infty)]^n\right\}. \tag{A-12.6} $$

*An explanation of the negative sign in the residue R_∞ is given by E. J. Townsend in reference [19], page 286.

For a liquid bottom comparison with equation (1-7.8) shows that

$$\lim_{u\to\infty} K^n = \lim_{u\to\infty} \left[\frac{\alpha_1 - b\alpha_2}{\alpha_1 + b\alpha_2}\right]^n = \left[\frac{1 - b}{1 + b}\right]^n, \qquad \text{(A-12.7)}$$

where the limit was taken directly after dividing numerator and denominator by u. Thus, for a liquid bottom

$$R_\infty = - \frac{(-1)^{n+m}}{\pi i\ R_m} \left[\frac{1 - b}{1 + b}\right]^n . \qquad \text{(A-12.8)}$$

For a rigid bottom L'Hopital's rule must be applied. The result is as follows:

$$[K(\infty)]^n = \lim_{u\to\infty} K^n \qquad \text{(A-12.9)}$$

$$= \lim_{u\to\infty} \left\{\frac{\frac{d}{du}\ (\text{numerator of } K)}{\frac{d}{du}\ (\text{denominator of } K)}\right\}^n = 1 .$$

Thus for a rigid bottom we obtain

$$R_\infty = -\frac{(-1)^{n+m}}{\pi i\ R_m} . \qquad \text{(A-12.10)}$$

A-13 STONLEY POLE RESIDUES FOR n = 1

It can be shown that

$$\lim_{z\to\pm ik} (z \pm ik)^n K^n$$

is finite whereas

$$\lim_{z\to\pm ik} (z \pm ik)^m K^n$$

is infinite for $m < n$. The Stonley poles are then poles of order n.

For $n = 1$ the Stonley poles are simple poles and may be evaluated as follows:

$$R_{ik} = \lim_{u\to ik} (u - ik)\,{}_nP'_m, \text{ and } R_{-ik} = \lim_{u\to -ik} (u + ik)\,{}_nP'_m \,. \tag{A-13.1}$$

It will be seen later that $R_{ik} = R_{-ik}$ so that we need only evaluate R_{ik}.

The rigid bottom reflection coefficient is

$$K = \frac{\alpha_1[(2u^2 + c_4^{-2})^2 - 4u^2\alpha_3\alpha_4] - b\alpha_3\ c_4^{-4}}{\alpha_1[(2u^2 + c_4^{-2})^2 - 4u^2\alpha_3\alpha_4] + b\alpha_3\ c_4^{-4}} \,. \tag{A-13.2}$$

Let $K = A/B$ where A and B are respectively the numerator and denominator of K. Then

$$R_{ik} = \left\{\lim_{u\to ik} \frac{\frac{1}{\pi i}(-1)^{m+1} A}{[u^2+c_1^{-2}]^{1/2}[u^2r^2+(\tau-d_m\alpha_1)^2]^{1/2}}\right\}\left\{\lim_{u\to ik} \frac{u(u - ik)}{B}\right\}. \tag{A-13.3}$$

Denote the first limit of R_{ik} by L_1 and the second limit by L_2. The limit L_1 is straightforward and becomes

$$L_1 = \frac{(-1)^{m+1}}{\pi i} \tag{A-13.4}$$

$$\frac{(c_1^{-2}-k^2)^{1/2}[(c_4^{-2}-2k^2)^2+4k^2(c_3^{-2}-k^2)^{1/2}(c_4^{-2}-k^2)^{1/2}]-b(c_3^{-2}-k^2)^{1/2}c_4^{-4}}{(c_1^{-2}-k^2)^{1/2}[-k^2r^2+\tau^2-2\tau d_m(c_1^{-2}-k^2)^{1/2}+d_m^2(c_1^{-2}-k^2)]^{1/2}} \,.$$

Using the Rosenbaum substitutions $g_1 = (k^2 - c_1^{-2})^{1/2}$, $g_3 = (k^2 - c_3^{-2})^{1/2}$, $g_4 = (k^2 - c_4^{-2})^{1/2}$, $\tau_m = \tau/R_m$, and $\cos \ell_m = d_m/R_m$, we have

$$L_1 = \frac{4(-1)^{m+1}}{\pi i} \frac{\left\{g_1\left[\left(\frac{c_4^{-2}}{2} - k^2\right)^2 - k^2 g_3 g_4\right] - \frac{bg_3}{4c_4^4}\right\}}{g_1 \left\{R_m^2\left[\tau_m^2 - (k^2 - c_1^{-2}\cos^2 \ell_m) - 2i\tau_m g_1 \cos \ell_m\right]\right\}^{1/2}} \tag{A-13.5}$$

Substituting $a = \tau_m^2 - (k^2 - c_1^{-2} \cos^2 \ell_m)$ and $f = 4\tau_m^2 g_1^2 \cos^2 \ell_m$ and removing the complex square root from the denominator, L_1 becomes

$$L_1 = \frac{4(-1)^{m+1}}{\pi i\ R_m} \frac{\left\{g_1\left[\left(\frac{c_4^{-2}}{2} - k^2\right)^2 - k^2 g_3 g_4\right] - \frac{bg_3}{4c_4^4}\right\}\left\{a + i\sqrt{f}\right\}^{1/2}}{g_1 (a^2 + f)^{1/2}} . \tag{A-13.6}$$

Looking ahead, L_2 will be imaginary; hence, only the imaginary part of $(a + i\sqrt{f})^{1/2}$ gives a real contribution to ${}_nP_m$. Using the definition of a complex square root and using the half-angle formula for a positive sine, we obtain

$$L_1 = \frac{4(-1)^{m+1}}{\pi\ R_m} \frac{\left\{g_1\left[\left(\frac{c_4^{-2}}{2} - k^2\right)^2 - k^2 g_3 g_4\right] - \frac{bg_3}{4c_4^4}\right\}\left\{\frac{\sqrt{2}}{2}\left[(a^2+f)^{1/2} - a\right]^{1/2}\right\}}{g_1 (a^2+f)^{1/2}} . \tag{A-13.7}$$

From equation (A-13.3), we can write the second limit as

$$L_2 = \lim_{u \to ik} \frac{u(u - ik)}{\alpha_1\left[(2u^2 + c_4^{-2})^2 - 4u^2\alpha_3\alpha_4\right] + b\alpha_3\ c_4^{-4}} . \tag{A-13.8}$$

Since ik is a root of the numerator and denominator, L_2 is of the indeterminant form 0/0. Applying L'Hopital's rule, we obtain

$$L_2 = \lim_{u \to ik} \frac{f(u)}{g(u)} = \lim_{u \to ik} \frac{f'(u)}{g'(u)} . \tag{A-13.9}$$

$$L_2 = \lim_{u \to ik} \tag{A-13.10}$$

$$\frac{2u - ik}{\frac{u}{\alpha_1}[(2u^2+c_4^{-2})^2-4u^2\alpha_3\alpha_4]+\alpha_1\left\{8u(2u^2+c_4^{-2})-8u\alpha_3\alpha_4-4u^3\left(\frac{\alpha_4}{\alpha_3}+\frac{\alpha_3}{\alpha_4}\right)\right\}+\frac{bu}{\alpha_3 \ c_4^{4}}}.$$

After carrying out the limit and substituting g_1, g_3, and g_4, we find

$$L_2 = \tag{A-13.11}$$

$$\frac{ik/4}{\frac{k}{g_1}\left[\left(\frac{c_4^{-2}}{2}-k^2\right)^2-k^2g_3g_4\right]-g_1k\left\{4\left(\frac{c_4^{-2}}{2}-k^2\right)+2g_3g_4+k^2\left(\frac{g_4}{g_3}+\frac{g_3}{g_4}\right)\right\}+\frac{bk}{4g_3 \ c_4^{4}}}.$$

Now since the factor k can be cancelled from numerator and denominator of L_2 leaving both L_1 and L_2 functions only of k^2, we have $R_{-ik} = R_{ik}$.

Combining equations (A-13.7) and (A-13.11) yields

$$R_{ik} + R_{-ik} = 2R_{ik} = \frac{\Delta_m}{\pi i} = (-1)^{m+1}\frac{i\sqrt{2}\,k}{\pi R_m g_1}\left\{\frac{(a^2+f)^{1/2}-a}{a^2+f}\right\}^{1/2}\Gamma, \tag{A-13.12}$$

where

$$\Gamma = \tag{A-13.13}$$

$$\frac{g_1\left[\left(\frac{c_4^{-2}}{2}-k^2\right)^2-k^2g_3g_4\right]-bg_3/(4\ c_4^{-4})}{\frac{k}{g_1}\left[\left(\frac{c_4^{-2}}{2}-k^2\right)^2-k^2g_3g_4\right]-g_1k\left\{4\left(\frac{c_4^{-2}}{2}-k^2\right)+2g_3g_4+k^2\left(\frac{g_4}{g_3}+\frac{g_3}{g_4}\right)\right\}+\frac{bk}{4g_3\ c_4^{4}}}.$$

Thus we have obtained the Stonley pole residue. The expression for ${}_{1}P_{m}$ can then be evaluated using the "method of residues". The branch line integrals and the complete solutions for n = 1 are obtained in Chapters II and IV. Because of the complexity of the Stonley pole residue, evaluation of ${}_{n}P_{m}$ for a rigid bottom when n > 1 using the "method of residues" is impractical. The method described in Chapter V which uses complex computer arithmetic must then be used.

APPENDIX B

CHARACTERISTIC TIMES IN REFLECTION PROBLEMS

B-1 INTRODUCTION

Many important features of the bottom reflection process are described simply and adequately by plane wave theory. The characteristic times of the reflection problem are among these. Snay [20] has given a discussion on the elements of the reflection process, and a few of his points are briefly treated here. The reader is referred to Snay's paper for further details. For simplicity our discussion is limited to a liquid bottom, but the results are also applicable to a rigid bottom. The discussion in Appendix E on the reflection coefficient gives further insight into the reflection process.

In the plane wave theory two types of reflection are distinguished: (a) regular or undercritical reflection and (b) total or supercritical reflection. These types of reflection depend on the angle of incidence θ_1 and the angle of refraction or transmission θ_2. The angle of reflection is the same as the incident angle. The angles are determined by rays perpendicular to the plane wave fronts as shown in Figure B-1.1 below. The angles θ_1 and θ_2 are related by Snell's law

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{c_1}{c_2} . \qquad \text{(B-1.1)}$$

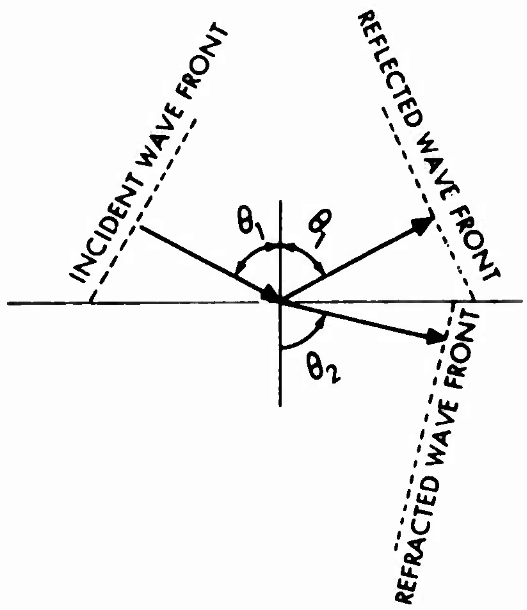

FIG. B-1.1 PLANE WAVE REFLECTION ANGLES

Assuming that $c_2 > c_1$, then the refracted angle θ_2 will be larger than the incident angle θ_1. When $\theta_2 = 90°$, the refracted ray lies along the interface and no transmission takes place into the lower medium. The corresponding incident angle is called the critical angle, θ_c,

$$\theta_c = \arcsin (c_1/c_2). \qquad \text{(B-1.2)}$$

For incident angles less than θ_c, undercritical reflection, an incident pressure history P(t) has a reflection K·P(t) and a refracted wave W·P(t) which have the same shape as the incident wave. The expressions K and W are respectively the time independent reflection and refraction coefficients. The incident energy is then divided between the reflected and transmitted waves.

For incident angles greater than θ_c, supercritical reflection, the refracted wave travels along the interface with the velocity

of the bottom. The refracted energy is then returned to the first medium so that there is no energy transmitted into the bottom - hence the expression "total reflection". In this case the reflected wave is not an image of the incident wave P(t) but is $P(t + T_s)$, the reflection coefficient being 1. The time T_s is the phase shift and is related to the angle of phase shift ϕ described in Appendix E.

The supercritically refracted wave is called the "precursor" since it arrives before the main reflected wave. Plane wave theory predicts an infinitely long precursor, but ray theory is useful in describing the precursor path and arrival time for a spherical wave and is used in Section B-2 below.

The figures B-1.2 and B-1.3 below are examples of the pressure history in which both the receiver and the exponentially decaying source are in the first medium. The second medium or bottom is a non-rigid material for which $c_2 > c_1$. The curve of Figure B-1.2 represents undercritical reflection, and the curve of Figure B-1.3 represents supercritical reflection.

In the regular or undercritical reflection the direct wave arrives first followed by the reflected wave which is an image of the direct wave. There is no precursor.

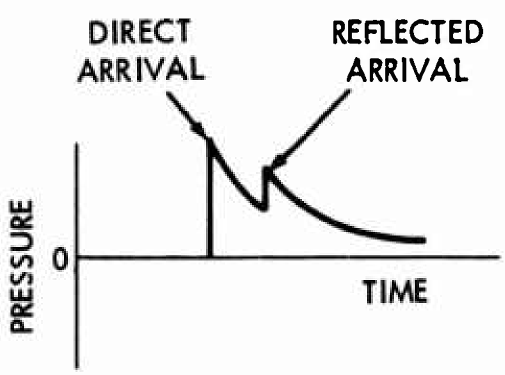

FIG. B-1.2 UNDERCRITICAL REFLECTION

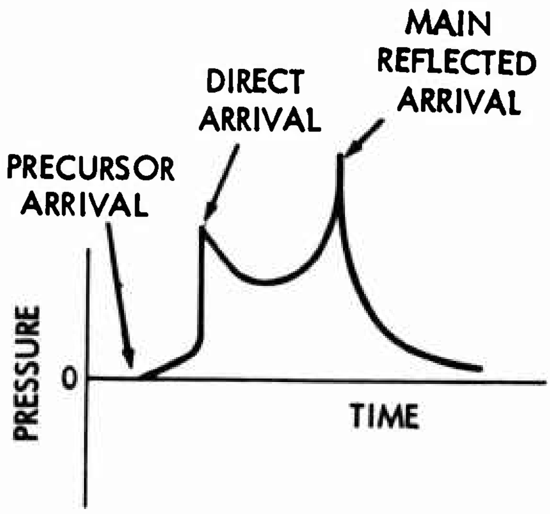

FIG. B-1.3 SUPERCRITICAL REFLECTION

In the supercritical case illustrated above the precursor arrives first followed by the direct wave and then the main reflection. In other cases the direct wave might arrive before the precursor, but in any case the precursor arrives before the main reflection. **The shape of the reflected wave is different from that of the direct wave.**

B-2 ARRIVAL TIMES OF DIRECT WAVE AND SIMPLE REFLECTION

In Figure B-2.1 below representative paths are shown for the direct wave, the main or simple reflection, and the precursor. From this diagram the derivation of the arrival times of the direct wave and the main reflection are trivial. They are as follows: The direct wave arrival time is

$$\tau_D = c_1^{-1} R , \tag{B-2.1}$$

where $R = (z^2 + r^2)^{1/2}$. The arrival time of the main reflection is

$$ {}_1\tau_1 = c_1^{-1} \, {}_1R_1 , \tag{B-2.2}$$

where ${}_1R_1 = [(2h - z)^2 + r^2]^{1/2}$.

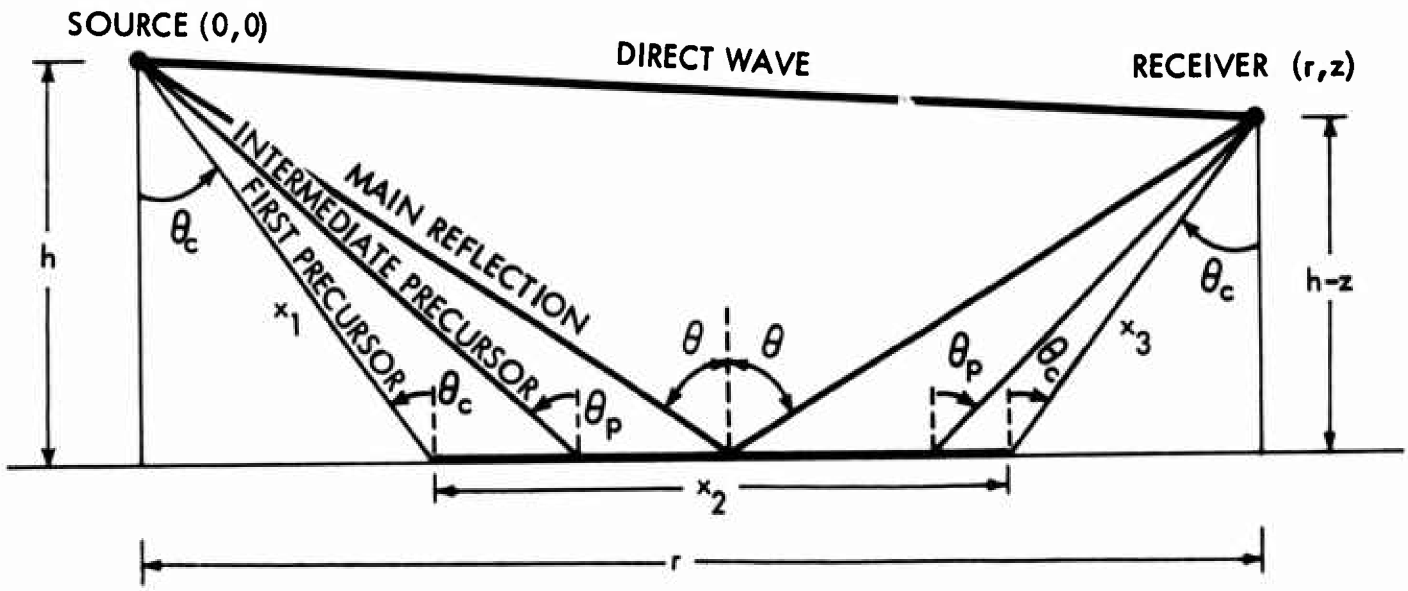

FIG. B-2.1 PATHS OF WAVES WITHOUT SURFACE REFLECTION

In the spherical wave problem the arrival time of the main reflected wave (n,m) is similarly

$$ {}_n\tau_m = c_1^{-1} \; {}_nR_m \, , \tag{B-2.3}$$

where ${}_nR_m = ({}_nd_m^2 + r^2)^{1/2}$.

The peak of the main reflection occurs at the instant the wave arrives at time $\tau = c_1^{-1} \; {}_nR_m$. The pressure ${}_nP_m$ at $\tau = c_1^{-1} \; {}_nR_m$ cannot be obtained directly from the equations of Chapters II, IV, and V, but must be obtained from equation (1-11.14) by a limiting process as $\tau \to c_1^{-1} \; {}_nR_m$. It has been shown (for example see Towne [21]) that for an angle of incidence θ less than the critical angle θ_c the peak of ${}_nP_m$ (in the units used in this paper) is given by

$$ {}_nP_m = K_p / {}_nR_m \text{ at } \tau = c_1^{-1}\ {}_nR_m , \tag{B-2.4} $$

where K_p is the plane wave reflection coefficient given by equations (E-4) and (E-8). But for angles of incidence above critical, ${}_nP_m$ tends logarithmically to infinity as $\tau \to c_1^{-1}\ {}_nR_m$.

The arrival time of the precursor is more complicated than those of the direct wave and the main reflection and is thus treated in more detail below.

B-3 ARRIVAL TIME OF THE PRECURSOR FROM RAY THEORY

Using ray theory the wave which we call the precursor can be visualized (see Figure B-2.1) as composed of rays which have incident angles θ_p between θ_c and θ. These waves travel along the interface with the velocity c_2 of the bottom and finally leave the bottom at the angle θ_p.

For the case $c_1 > c_2$ there is no real critical angle and no precursor. A wave traveling along the interface with the slower velocity of the bottom arrives after the simple reflection.

The path of the precursor wave shown in Figure B-2.1 corresponds to the precursor wave n = m = 1 of our spherical wave solution. The first response of the precursor travels along the path with incident angle θ_c. The arrival time $\delta = {}_1\delta_1$ of the precursor wave for n = m = 1 is then

$$ \delta = c_1^{-1}\ (x_1 + x_3) + c_2^{-1} x_2 . \tag{B-3.1} $$

From Snell's law we have $\sin\theta_c = c_1/c_2 = c_2^{-1}/c_1^{-1}$; hence

$$\cos\theta_c = (c_1^{-2} - c_2^{-2})^{1/2}/c_1^{-1}. \tag{B-3.2}$$

Now $x_1 = h\sec\theta_c$, $x_2 = r - h\tan\theta_c - (h-z)\tan\theta_c$, and $x_3 = (h-z)\sec\theta_c$. Then

$$\delta = c_1^{-1}(2h - z)\sec\theta_c + c_2^{-1}[r - (2h - z)\tan\theta_c]. \tag{B-3.3}$$

Substitution of expressions for $\sec\theta_c$ and $\tan\theta_c$ yields

$$\delta = \frac{c_1^{-2}(2h - z)}{(c_1^{-2} - c_2^{-2})^{1/2}} + c_2^{-1}\left[r - \frac{(2h - z)c_1^{-1}}{(c_1^{-2} - c_2^{-2})^{1/2}}\right] \tag{B-3.4}$$

$$\delta = r\,c_2^{-1} + (2h - z)(c_1^{-2} - c_2^{-2})^{1/2}. \tag{B-3.5}$$

In this spherical wave notation the height from the image source to the receiver is ${}_1d_1 = 2h - z$ so that ${}_1R_1 = ({}_1d_1^2 + r^2)^{1/2}$ and

$$\delta = r\,c_2^{-1} + {}_1d_1\,(c_1^{-2} - c_2^{-2})^{1/2}. \tag{B-3.6}$$

With $\sin\ell_m = r/{}_1R_1$, $\cos\ell_m = {}_1d_1/{}_1R_1$, and $\delta_m = \delta/{}_1R_1$ we can write

$$\delta_m = c_2^{-1}\sin\ell_m + (c_1^{-2} - c_2^{-2})^{1/2}\cos\ell_m. \tag{B-3.7}$$

The above equations give δ for the pressure response ${}_1P_1$. The precursor arrival time ${}_n\delta_m$ for ${}_nP_m$ can similarly be derived using ray theory and can be written

$$ {}_n\delta_m = r\, c_2^{-1} + {}_n d_m \,(c_1^{-2} - c_2^{-2})^{1/2}. \qquad \text{(B-3.8)}$$

The above derivation of ${}_n\delta_m$ uses only geometrical ray considerations. We will show below that the same equation is obtained from equation (1-11.14) for ${}_nP_m$. Hence geometrical ray theory correctly determines the arrival times of not only the direct wave and the main reflection, but also the precursor.

B-4 ARRIVAL TIME OF THE PRECURSOR FROM THE EQUATION FOR ${}_nP_m$

Aside from ray theory, the arrival time ${}_n\delta_m$ is interpreted as the time for which ${}_nP_m$ first becomes non-zero. Since ray theory predicts there is no precursor if $c_1 > c_2$, we will first assume that $c_2 > c_1$ and also that $\tau < c_1^{-1}\,{}_nR_m$ and $c_2^{-1} < |u_1| < c_1^{-1}$. The integration path is then that shown in Figure B-4.1 below. The placing of u_1 below the real axis and $\tilde{u}_1$ above the real axis can be shown to be correct by using the result that ${}_nP_m = 0$ for $\tau < c_1^{-1}\,{}_nd_m$. The choice of $|u_1| < c_1^{-1}$ is not arbitrary. In Appendix A it is shown that when $\tau < c_1^{-1}\,{}_nR_m$ then $|u_1| < c_1^{-1}$.

It must be remembered that $\pm ic_1^{-1}$ and $\pm ic_2^{-1}$ are respectively branch points of $\alpha_1 = (u^2 + c_1^{-2})^{1/2}$ and $\alpha_2 = (u^2 + c_2^{-2})^{1/2}$. Also u_1 and $\tilde{u}_1$ are branch points of $\gamma = \left\{u^2r^2 + [\tau - {}_nd_m(c_1^{-2} + u^2)^{1/2}]^2\right\}^{1/2}$.

When $\tau < c_1^{-1}\,{}_nR_m$ the square root $(\tau - c_1^{-1}\,{}_nR_m)^{1/2}$ is imaginary so that u_1 and $\tilde{u}_1$ are also imaginary. Thus the integration path is taken arbitrarily close to the imaginary axis in the right half plane. The end points u_1 and $\tilde{u}_1$ are singularities of the integrand, γ being zero at these points. However, the integral converges and the integration can be carried from u_1 to $\tilde{u}_1$.

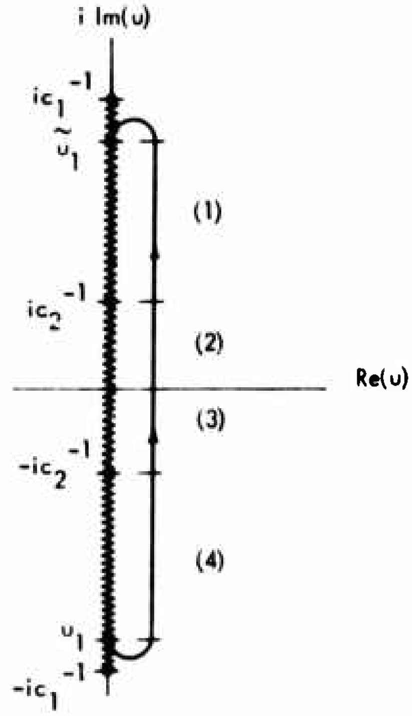

FIG. B-4.1 INTEGRATION PATH IN THE u-PLANE FOR $\tau < c_1^{-1}\,{}_nR_m$ AND $c_2 > c_1$

As the path approaches the imaginary axis, the end paths vanish and make no contribution to the integral. The square roots α_1, α_2, and γ are all real and positive on paths (2) and (3) since $|u| \leq c_1^{-1}$ and $|u| \leq c_2^{-1}$. The change in variables $u = ix$, for real x, in equation (1-11.14) for ${}_nP_m$ yields for the contribution I along paths (2) and (3)

(B-4.1)

$$I = -(-1)^{n+m}\frac{1}{\pi i}\int_{-c_2^{-1}}^{c_2^{-1}} x\alpha_1^{-1}\, K^n\left[-x^2r^2+(\tau-{}_nd_m\alpha_1)^2\right]^{-1/2}dx,$$

where α_1, α_2, K^n, and γ are now real functions of x^2. The integral is a real function of x^2 only and is then zero between the limits $-c_2^{-1}$ and c_2^{-1}. A non-zero ${}_nP_m$ can then result only from the contributions of paths (1) and (4). When $c_1 > c_2$ we have $|u_1| < c_2^{-1}$ and hence ${}_nP_m = 0$ for all $\tau < c_1^{-1}\,{}_nR_m$.

From the above conclusions we can now determine ${}_n\delta_m$. For $c_1 > c_2$ the pressure ${}_nP_m$ can be non-zero only if $\tau \geq c_1^{-1}\,{}_nR_m$. Thus for this case ${}_n\delta_m = c_1^{-1}\,{}_nR_m$ and there is no precursor. For $c_1 < c_2$ the pressure can be non-zero only if $|u_1| > c_2^{-1}$. Thus ${}_n\delta_m$ is the time τ for which $|u_1| = c_2^{-1}$. Using equation (1-10.2) which defines u_1 and replacing τ by ${}_n\delta_m$ in $|u_1|$ yields

$${}_nR_m^{-2}\left[{}_n\delta_m\, r - {}_nd_m(c_1^{-2}\,{}_nR_m^2 - {}_n\delta_m^{\,2})^{1/2}\right] = c_2^{-1}. \qquad \text{(B-4.2)}$$

Solving for ${}_n\delta_m$, we get

$${}_n\delta_m = r\, c_2^{-1} + {}_nd_m(c_1^{-2} - c_2^{-2})^{1/2}. \qquad \text{(B-4.3)}$$

Thus the same precursor arrival time ${}_n\delta_m$ is obtained from ray theory as from the spherical wave relection theory.

Above we made no mention of the critical angle of incidence θ_c, hence we must determine its meaning in ${}_n\delta_m$.

The angle of incidence θ of the main reflected ray is given by $\theta = \arcsin(r/{}_nR_m)$. When $\theta = \theta_c$ we have $\sin\theta = c_1/c_2$ and ${}_n\delta_m$ reduces to ${}_n\delta_m = c_1^{-1}\,{}_nR_m$ the arrival time of the main reflection. Hence, if θ is less than θ_c, the critically refracted wave arrives after the main reflection.

Summarizing our results, the arrival time ${}_n\delta_m$ of the first arriving wave can be expressed by the following equations:

$${}_n\delta_m = r\, c_2^{-1} + {}_nd_m(c_1^{-2} - c_2^{-2})^{1/2} \text{ for } \theta \geq \theta_c \qquad \text{(B-4.4)}$$

and

$$_n\delta_m = c_1^{-1}\,_nR_m \text{ for } \theta < \theta_c. \qquad \text{(B-4.5)}$$

The above equation is also valid for a rigid bottom where there is an additional precursor wave, the shear precursor. This wave is similar to the above precursor except it propagates along the interface as a shear wave with velocity c_4. The critical angle for the shear wave and the arrival time of the shear precursor are obtained by replacing c_3 by c_4 in the appropriate expressions.

A typical pressure history for an exponentially decaying pulse measured and produced in a liquid layer over a rigid bottom is shown below in Figure B-4.2. The case shown is for $c_3 > c_4 > c_1$ and for supercritical reflection of both sound and shear waves, where c_3 denotes the sound velocity in the rigid bottom.

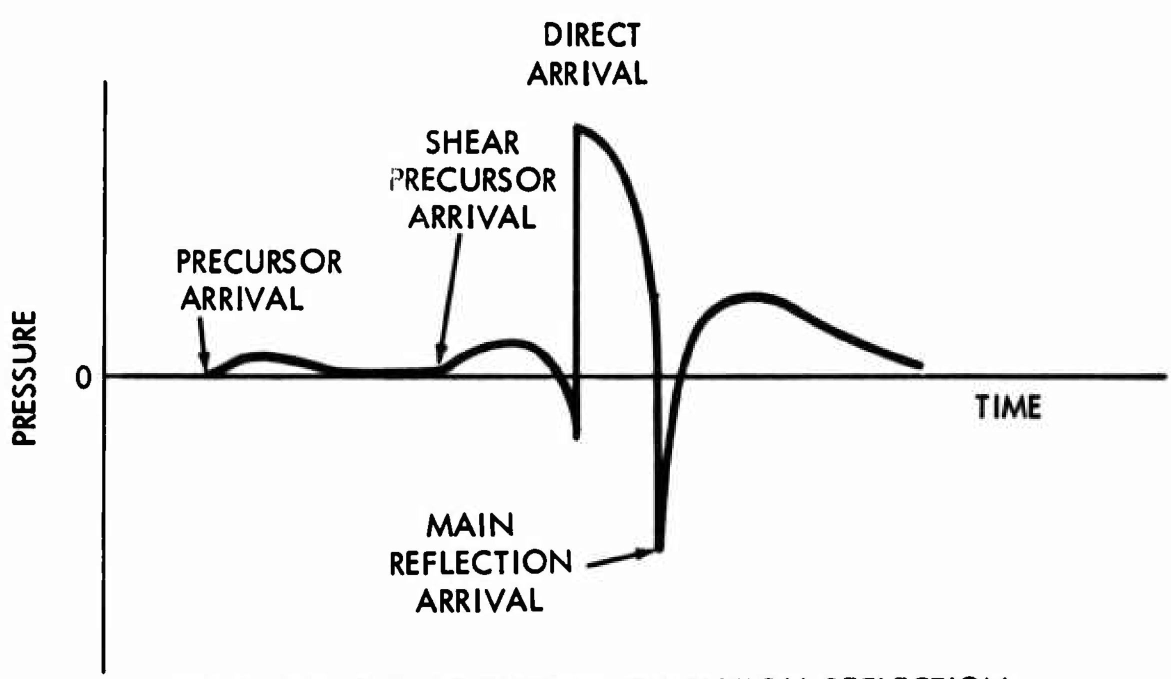

FIG. B-4.2 RIGID BOTTOM SUPERCRITICAL REFLECTION

The main reflection is negative for this case. This inversion of the peak results when the phase shift ϕ has values above 180°.

Above, we have determined the arrival times of the direct wave, precursor waves, and the simple reflection. These times are the characteristic times of the reflection problem and are important considerations when a pressure-time history is constructed.

APPENDIX C

REPLACING THE INTEGRATION PATH C_w IN THE COMPLEX τ-PLANE BY A PATH ALONG THE REAL AXIS

In Section 1-8 after the change in integration variable from u to τ it was stated that the integration path can be taken along the real axis. But the change of variables (u, τ) does not lead directly to this path, and it is the object of this appendix to show that the value of the integral is unchanged under this change of path.

The change in variables from u to τ at constant w is defined in Section 1-8 by the equation (1-8.1), namely

$$\tau = \alpha_1 \, {}_n d_m + iur \cos w, \qquad \text{(C-1)}$$

where $\alpha_1 = (u^2 + c_1^{-2})^{1/2}$. The integral in equation (1-7.5) becomes

$${}_n\bar{P}_m = (-1)^{n+m} \mathrm{Re}\left\{\frac{2}{\pi}\int_0^{\pi/2}\left[\oint_{C_w, \; c_1^{-1}\,{}_n d_m}^{\infty} u\alpha_1^{-1} K^n \left(\frac{\partial u}{\partial \tau}\right)_w \exp(-s\tau)\, d\tau\right] dw\right\}. \qquad \text{(C-2)}$$

In this equation τ is a complex variable, and the integration is carried out on the path C_w, the exact path depending on w. The path C_w is shown in Figure C-1 as the heavy line which begins at the point $\tau = c_1^{-1}\,{}_n d_m$ on the real axis and extends to infinity in the first quadrant.* The letters A and B denote points at infinity.

*For further discussion on these integration paths see Cagniard [22].

We propose to use Cauchy's theorem to make the path change. Thus we must know the nature and location of the singularities of the integrand. Before working with the path C_w, we must then examine the partial derivative $(\partial u/\partial \tau)_w$ for singularities.

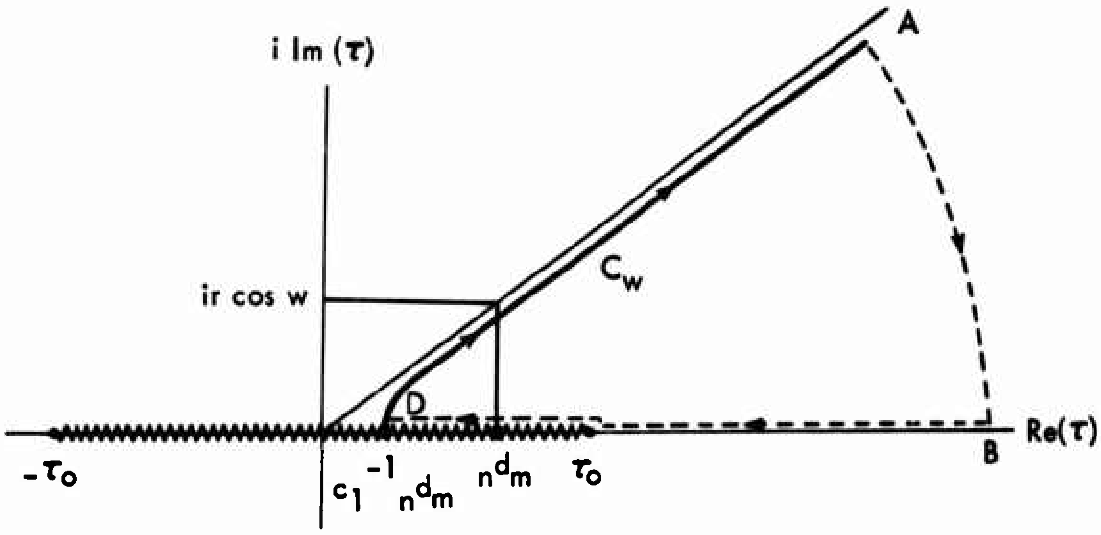

FIG. C-1 INTEGRATION PATH IN THE τ-PLANE

From equation (C-1) we obtain the partial derivative

$$\left(\frac{\partial u}{\partial \tau}\right)_w = \left[ir \cos w + u\,{}_n d_m/\alpha_1\right]^{-1}. \qquad \text{(C-3)}$$

After eliminating u, we obtain

$$\left(\frac{\partial u}{\partial \tau}\right)_w = \frac{\tau\,{}_n d_m + ir \cos w\,[\tau^2 - c_1^{-2}({}_n d_m^2 + r^2\cos^2 w)]^{1/2}}{[{}_n d_m^2 + r^2\cos^2 w][\tau^2 - c_1^{-2}({}_n d_m^2 + r^2\cos^2 w)]^{1/2}} . \qquad \text{(C-4)}$$

The partial derivative $(\partial u/\partial \tau)_w$ is finite on C_w for all w except $w = \pi/2$. When $w = \pi/2$ the partial derivative approaches infinity

at $\tau = c_1^{-1}\,{}_n d_m$. But the integrand remains finite since at these values of τ and w, u is zero and the limit $\lim_{\tau \to c_1^{-1}\,{}_n d_m}\left[u\left(\frac{\partial u}{\partial \tau}\right)_w\right]$ is finite.

It is very difficult to integrate along the path C_w; hence the shift of the integration to the real axis is necessary. The shift can be performed using Cauchy's theorem if we close the path C_w with the infinite arc A B and the path B τ_0 D along the real axis. The square roots in $(\partial u/\partial \tau)_w$ can be made analytic by making a branch cut from $-\tau_0$ to τ_0 along the real axis, where $\tau_0 = c_1^{-1}({}_n d_m^2 + r^2\cos^2 w)^{\frac{1}{2}}$. The integration path from τ_0 to D is then taken above, but infinitely near, the cut. The integrand, then, has no singularities within or on the closed path $DAB\tau_0D$. If we denote the integrand by $f(\tau)$, then by Cauchy's theorem we obtain

$$\oint_{DAB\tau_0 D} f(\tau)d\tau = \oint_w f(\tau)d\tau + \int_A^B f(\tau)d\tau + \int_B^D f(\tau)d\tau = 0. \quad \text{(C-5)}$$

The arc A B at infinity makes no contribution since the integrand is identically zero there. Then the integrals reduce to

$$\oint_w f(\tau)d\tau = -\int_\infty^{c_1^{-1}\,{}_n d_m} f(\tau)d\tau = \int_{c_1^{-1}\,{}_n d_m}^\infty f(\tau)d\tau. \quad \text{(C-6)}$$

The integration for ${}_n P_m$ is now performed along the real axis, and equation (C-2) reduces to

$${}_n\bar{P}_m = (-1)^{n+m}\mathrm{Re}\left\{\frac{2}{\pi}\int_0^{\pi/2}\int_{c_1^{-1}\,{}_n d_m}^\infty u\alpha_1^{-1}K^n\left(\frac{\partial u}{\partial \tau}\right)_w \exp(-s\tau)\,d\tau dw\right\}. \quad \text{(C-7)}$$

This is the desired equation since it allows us to identify the second integral as a one sided Laplace transform.

APPENDIX D

CONVOLUTION INTEGRAL FOR A STEP WAVE SOURCE

The convolution integral which allows us to obtain pressure-time histories from step wave solutions can be derived from the convolution integral theorem of Laplace transforms. This theorem states: If $\bar{q}$, $\bar{f}$, and $\bar{g}$ are the Laplace transforms with respect to τ of the functions q, f, and g such that $\bar{q} = \bar{f} \cdot \bar{g}$, then we have

$$q = \int_0^{\tau} f(v)g(\tau - v)dv = \int_0^{\tau} f(\tau - v)g(v)dv. \qquad \text{(D-1)}$$

The transform $\bar{P}_i$ of the pressure response P_i to an arbitrary source $P_d = F(\tau - c_1^{-1}R)/R$ can be expressed

$$\bar{P}_i = s\bar{F}(s)\ \bar{A}_i(r,z,s), \qquad \text{(D-2)}$$

where $\bar{A}_i(r,z,s)$ is a solution of the transformed wave equation (1-3.4). The advantage of writing $\bar{P}_i$ in this form can be seen by employing the convolution theorem. The expression $s\bar{F}(s)$ is the transform of the derivative $F'(\tau)$ if $F(0) = 0$. Thus the application of equation (D-1) yields

$$P_i = \int_0^{\tau} F'(\tau - v)\ A_i(r,z,v)dv. \qquad \text{(D-3)}$$

To find the meaning of $A_i(r,z,\tau)$, set $F(\tau) = H(\tau)$ then the derivative is $F'(\tau) = \delta(\tau)$, where $\delta(\tau)$ is the Delta Function* defined $\delta(\tau) = 0$ for $\tau \neq 0$ and

$$\int_0^\infty \delta(\tau)d\tau = 1.$$

The pressure response P_i becomes

$$P_i = \int_0^\tau \delta(\tau - v)\ A_i(r,z,v)dv = A_i(r,z,\tau). \qquad \text{(D-4)}$$

The function $A_i(r,z,\tau)$ is thus the pressure response to a source $H(\tau - c_1^{-1}R)/R$. The value of the substitution, equation (D-2), is evident. Once solutions $A_i(r,z,\tau)$ for a source $P_A = H(\tau - c_1^{-1}R)/R$ have been obtained, then solutions for a source $P_d = F(\tau - c_1^{-1}R)/R$ can be obtained by using equation (D-3).

As an example, consider the exponential pulse

$$F(\tau - c_1^{-1}R) = P_0 H(\tau - c_1^{-1}R)\ \exp[-(\tau - c_1^{-1}R)/\theta], \qquad \text{(D-5)}$$

where P_0 is the peak pressure at unit distance from the origin and θ is the time decay constant. Then we obtain

$$F'(\tau - v) = -\frac{P_0}{\theta} H(\tau - v)\ \exp[-(\tau - v)/\theta] + \delta(\tau - v)\ P_0 \exp[-(\tau - v)/\theta] \qquad \text{(D-6)}$$

---

*Good discussions that make the delta function plausible are given by Mackie [23] pp. 13,14 and Churchill [24] pp. 26,27.

and

$$P_i(\tau) = P_0\left\{A_i(r,z,\tau) - \frac{1}{\theta}\int_0^\tau H(\tau - v)\exp[-(\tau - v)/\theta]\, A_i(r,z,v)\,dv\right\}. \tag{D-7}$$

The pressure-time history for an exponential pulse source may then be obtained from the step wave solution.

The step wave solution in the water $A_1(r,z,\tau)$ may be represented using the notation introduced in Sections 1-4, 1-5, and 1-6 as the following sum:

$$A_1(r,z,\tau) = {}_0P_1 + {}_0P_2 + \sum_{m=1}^{4}\sum_{n=1}^{\infty} {}_nP_m, \tag{D-8}$$

where ${}_0P_1$ and ${}_0P_2$ are respectively the surface reflection and direct wave for the step wave source and are derived in Appendix F. At any time τ only those ${}_nP_m$ for which $\tau > {}_n\delta_m$ are non-zero. Applying the convolution integral, equation (D-3), to equation (D-8), we obtain the total pressure $P_1(\tau)$ in the water for a source $P_d = F(\tau - c_1^{-1}R)/R$:

$$P_1(\tau) = P_s(\tau) + P_d(\tau) + \sum_{m=1}^{4}\sum_{n=1}^{\infty}\left\{\int_0^\tau F'(\tau - v)\,{}_nP_m(v)\,dv\right\}, \tag{D-9}$$

where P_s and P_d are respectively the surface reflection and the direct wave. Thus the solution in the water for an arbitrary source P_d may be obtained from the sum of step wave solutions ${}_nP_m$.

APPENDIX E

THE PLANE WAVE REFLECTION COEFFICIENT IN THE SPHERICAL WAVE SOLUTION

The reflection coefficient K obtained in the spherical wave solution can be identified as the plane wave reflection coefficient K_p for some values of the integration variable u. The rigid bottom reflection coefficient K, equation (3-6.2), is

$$K = \frac{\alpha_1[(2u^2 + c_4^{-2})^2 - 4u^2\alpha_3\alpha_4] - b\alpha_2 c_4^{-4}}{\alpha_1[(2u^2 + c_4^{-2})^2 - 4u^2\alpha_3\alpha_4] + b\alpha_3 c_4^{-4}} . \qquad \text{(E-1)}$$

For a liquid bottom $c_4 = 0$, and c_2 and α_2 replace c_3 and α_3. The above equation then reduces to the liquid bottom reflection coefficient

$$K = \frac{\alpha_1 - b\alpha_2}{\alpha_1 + b\alpha_2} . \qquad \text{(E-2)}$$

The variable u can be expressed

$$u = ic_1^{-1} \sin\theta , \qquad \text{(E-3)}$$

where θ can be interpreted as the angle of incidence of a plane wave for imaginary u for which $|u| \leq c_1^{-1}$. Substitution of this expression for u in K yields the plane wave reflection coefficient K_p which can be put in the following form given by Snay [20]:

$$K_p = \frac{K_1 + K_2 - K_3}{K_1 + K_2 + K_3} , \qquad \text{(E-4)}$$

where

$$K_1 = \cos\theta \, [1 - 2(c_4/c_1)^2 \sin^2\theta]^2 \qquad \text{(E-5)}$$

$$K_2 = 4(c_4/c_1)^3 \sin^2\theta \cos\theta \sqrt{1 - (c_4/c_1)^2\sin^2\theta} \, \sqrt{1 - (c_3/c_1)^2\sin^2\theta} \qquad \text{(E-6)}$$

$$K_3 = (\rho_1 c_1/\rho_2 c_3) \sqrt{1 - (c_3/c_1)^2\sin^2\theta} \, . \qquad \text{(E-7)}$$

For a liquid bottom we have $c_4 = 0$ and $c_2 = c_3$. The above expression of K_p reduces to

$$K_p = \frac{\cos\theta - K_3}{\cos\theta + K_3} \, . \qquad \text{(E-8)}$$

The plane wave reflection coefficient K_p given by equations (E-4) and (E-8) is real with absolute value less than one if $\theta < \theta_{cr} = \arcsin (c_1/c_3)$. For supercritical θ ($\theta \geq \theta_{cr}$) the reflection coefficient is complex since the square root $\sqrt{1 - (c_3/c_1)^2\sin^2\theta}$ is imaginary. The coefficient K_p can then be represented in the complex plane as shown in Figure E-1 below. The angle ϕ is called the "phase shift" and is defined by

$$\tan\phi = - \frac{\mathrm{Im}(K_p)}{\mathrm{Re}(K_p)} \, . \qquad \text{(E-9)}$$

The negative sign in the phase shift is chosen, when the square roots are taken to be positive, so that the plane wave solution will remain finite as the depth z below the source approaches infinity.

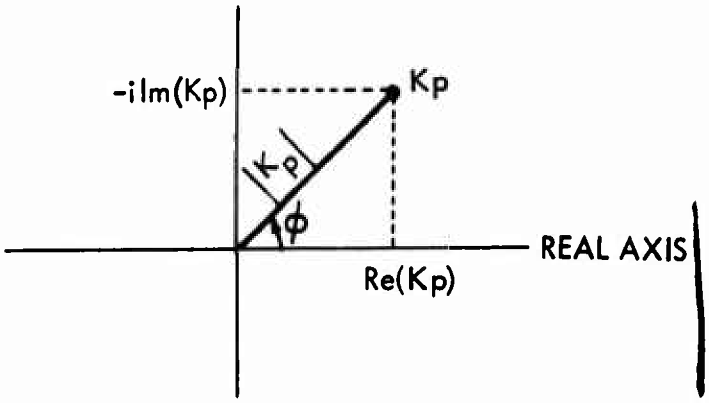

FIG. E-1 SUPERCRITICAL PLANE WAVE REFLECTION COEFFICIENT

For a liquid bottom when $\theta \geq \theta_{cr}$ the absolute value or modulus of K_p is one, and the phase shift increases from 0° at $\theta = \theta_{cr}$ to 180° at $\theta = 90°$. However, for a rigid bottom a second critical angle, $\theta_{crs} = \arcsin(c_1/c_4)$, which is greater than θ_{cr} affects the phase shift and modulus of K_p. For $\theta = \theta_{cr}$ the coefficient K_p has a modulus of one and a zero phase shift. In the range $\theta_{cr} < \theta < \theta_{crs}$, $|K_p| < 1$ and the phase shift takes on values $0 \leq \phi \leq 180°$. When $\theta \geq \theta_{crs}$ the modulus $|K_p|$ is again one and the phase shift ϕ increases to a maximum value $180° < \phi_{max} < 360°$. Depending on the velocities of sound and shear propagation and densities of the two media, the phase shift ϕ can increase to values well over 180° in contrast to the liquid bottom which has a phase shift of at most 180°.

A simple interpretation of u in terms of θ can be made in the precursor integral for ${}_nP_m$ along the integration path shown in Figure 2-1.1. From equations (1-11.16) and (2-1.5) the change of variables $u = ix$ can be seen to yield the following form of the precursor integral:

$$ {}_nP_m = -(-1)^{n+m} \frac{2}{\pi i} \int_{c_2^{-1}}^{|\widetilde{u}_1|} \frac{x}{\alpha_1} \frac{K^n}{\gamma}\, dx, \tag{E-10}$$

where α_1, γ, and K^n are now functions of x and $|\widetilde{u}_1| = {}_nR_m^{-2}\,[\tau r - {}_nd_m\,(c_1^{-2}\,{}_nR_m^2 - \tau^2)^{1/2}]$. The change of variables $x = c_1^{-1} \sin\theta$ yields

$$ {}_nP_m = \int_{\theta_{cr}}^{\theta_1} f(\theta)\, d\theta , \tag{E-11}$$

where $\theta_1 = \arcsin\,(c_1|\widetilde{u}_1|)$. The angle θ_1 increases from θ_{cr} when $\tau = {}_n\delta_m$, the arrival time of the precursor, to the "incident angle" of the spherical wave, $\ell_m = \arcsin(r/{}_nR_m)$, when $\tau = c_1^{-1}\,{}_nR_m$.

The spherical wave precursor integral can then be interpreted as follows: The precursor wave is a superposition of rays with angles of incidence between θ_{cr} and θ_1. The angle θ_1 is the incident angle for the slowest ray path along which a wave can arrive at the receiver at time τ. As τ approaches $c_1^{-1}\,{}_nR_m$, the band of rays in the precursor integral takes on all values of θ between θ_{cr} and ℓ_m.

The main reflection ($\tau > c_1^{-1}\,{}_nR_m$) does not lend itself to the simple interpretation used above since the relation between u and θ requires complex values of θ.

In summary, the reflection coefficients $K(u)$ and $K_p(\theta)$ are equivalent, and the spherical wave solution can be visualized as a superposition of plane waves with angles of incidence θ which are real for $\tau < c_1^{-1}\,{}_nR_m$ and complex for $\tau > c_1^{-1}\,{}_nR_m$.

APPENDIX F

DIRECT WAVE AND SURFACE REFLECTION FROM STEP WAVE SOLUTIONS

The surface reflection and the direct wave for a step wave source are respectively represented in the general solution ${}_nP_m$ by the terms ${}_0P_1$ and ${}_0P_2$ where ${}_0d_1$ and ${}_0d_2$ are given by equations (1-6.5) and (1-6.6). Since the exponent n on the reflection coefficient K is zero, the equation (1-11.14) for ${}_0P_m$ reduces to

$$
\begin{aligned}
{}_0P_m &= 0 \qquad \tau < {}_0\delta_m \\
{}_0P_m &= \frac{(-1)^m}{\pi i}\int_{u_1}^{\tilde{u}_1} u\alpha_1^{-1}\,[u^2r^2 + (\tau - {}_0d_m\alpha_1)^2]^{-1/2}\,du \qquad \tau > {}_n\delta_m .
\end{aligned}
\tag{F-1}
$$

The waves ${}_0P_m$ do not impact on the bottom, and the square roots α_2 or α_3 and α_4 do not appear in the integrand. The branch line integrals between $-ic_1^{-1}$ and ic_1^{-1} or between u_1 and $\tilde{u}_1$ along the imaginary axis are zero. Consequently, ${}_0P_m$ is zero for $\tau \le c_1^{-1}\,{}_0R_m$.

The solution ${}_0P_m$ for $\tau > c_1^{-1}\,{}_0R_m$ is easily obtained using the "method of residues" explained in Appendix A. Equation (A-11.5) expresses ${}_nP_m$ as the sum of a branch line integral, and contributions from the Stonley poles and the pole at infinity. In the present case, ${}_0P_m$, the reflection coefficient is $K^\circ = 1$. Since the square roots α_1, α_3, and α_4 are not contained in K°, the branch line integral is zero and there are no Stonley poles. Hence, the only contribution to ${}_0P_m$ comes from the singularity at infinity, and we

can write

$$
\begin{aligned}
{}_{o}P_m &= 0 \qquad\qquad \tau \le c_1^{-1}\,{}_{o}R_m \\
{}_{o}P_m &= -\pi i\,R_\infty \qquad \tau > c_1^{-1}\,{}_{o}R_m
\end{aligned}
\qquad \text{(F-2)}
$$

Since $K^\circ = 1$ in the integrand of ${}_{o}P_m$, the residue at infinity R_∞ given by equation (A-12.8) or by equation (A-12.10) reduces to

$$R_\infty = \frac{-(-1)^m}{\pi i\,{}_{o}R_m}\,. \qquad \text{(F-3)}$$

The solution for ${}_{o}P_m$ is then

$$
\begin{aligned}
{}_{c}P_m &= 0 \qquad\qquad \tau < c_1^{-1}\,{}_{o}R_m \\
{}_{o}P_m &= \frac{(-1)^m}{{}_{o}R_m} \qquad \tau > c_1^{-1}\,{}_{o}R_m\,.
\end{aligned}
\qquad \text{(F-4)}
$$

The term ${}_{o}P_2$ is identified as the direct wave and can be written

$${}_{o}P_2 = P_A = H(\tau - c_1^{-1}\,R)/R\,, \qquad \text{(F-5)}$$

where H is the Heaviside step function and $R = {}_{o}R_2 = (z^2 + r^2)^{1/2}$. The above equation for P_A is identical to equation (1-4.11). Thus the variable τ defined by equation (1-8.1) is indeed the time τ.

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

U. S. Naval Ordnance Laboratory
White Oak, Silver Spring, Md. 20910

2a. REPORT SECURITY CLASSIFICATION: Unclassified

2b. GROUP

3. REPORT TITLE

Linear Theory of Bottom Reflections

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)

James R. Britt

6. REPORT DATE	7a. TOTAL NO OF PAGES	7b. NO OF REFS
12 May 1969	149	24

8a. CONTRACT OR GRANT NO

b. PROJECT NO ORDTASK ORD-035-200/092-1/UF17-553-502

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S): 69-44

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale, its distribution is unlimited.

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

Defense Atomic Support Agency
Naval Ordnance Systems Command

13. ABSTRACT

This paper gives a detailed mathematical development of a linear, spherical wave theory for the reflection of underwater explosion shock waves from plane bottoms of either fluid or rigid materials. The Laplace transform method of L. Cagniard is used to obtain integral solutions for the pressure which can be evaluated numerically.

The paper begins with linear equations of motion and proceeds in steps through the derivation of the wave equations and finally to solutions of the wave equations. Two methods of integrating the integral solution are discussed. First, the real part of the integral is separated from the imaginary part to allow integration using real arithmetic. Second, a method of integration using complex arithmetic is described. To aid the readers' understanding of the problem many important details are included in both the text and the appendices.

UNCLASSIFIED

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Underwater Explosions Nuclear Explosions Shock Waves Bottom Reflection of Shock Waves Acoustic Reflection Cagniard's Method						

DD FORM 1 NOV 65 1473 (BACK)

(PAGE 2)

UNCLASSIFIED

Security Classification